THE SECOND VIRIAL COEFFICIENT FOR CONTACT INTERACTING ANYONS

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ABSTRACT

The second virial coefficient is obtained for a Bose-Einstein gas of anyons in the presence of contact interaction. Particular care is devoted to the regularization methods, which are necessary to compute finite physical quantities in closed form. The harmonic potential well regularization appears to be viable in principle, quite natural from the physical point of view, but unmanageable from the technical point of view. On the contrary, dimensional regularization actually drives to the explicit result although less intuitive from a physical point of view.

1. Introduction

Theories of anyons, or particles with arbitrary spin and statistics in *2+1* dimensions, have been attracting considerable attention [1,2], due to their potential applications to the fractional quantum Hall effect [3], to high-T_c superconductivity and to the description of interacting cosmic strings [4] (for a review see [5]). Since the original analysis performed by Arovas et al. [2], a number of attempts have been made in order to obtain the thermodynamic properties of a gas of free anyons [6-9]. Thanks to its complete solvability, the system of two anyons has represented the "theoretical laboratory" where all the main characteristics of fractional statistics have been investigated.

In particular several methods have been developed in order to calculate the second virial coefficient (SVC) in the last few years [6-9]. As a matter of fact all of them imply the use of a regularization procedure, which appears unvoidable owing to the long range nature of the statistical potential (actually an Aharonov-Bohm (AB) potential [10]). More recently it has been realized that anyons, as free non relativistic particles, may feel the so called contact interaction [11]. As it is well known, in two and three dimensions δ-interactions are too "strong" and have to be regularized or, from a more sophisticated mathematical point of view, the self-adjoint extensions of the Hamiltonian operator must be considered [12] (for an exhaustive review see [13]). It is also known that they are nontrivial only when of attractive nature and that they generally break scale invariance, in spite of the fact that the classical Hamiltonian does indeed fulfil it [14].

In this paper we shall deal with the problem of computing the SVC for the anyon gas in the presence of contact interactions. After a critical overview on the known results, in sect. 2 we will prove that regularization procedures available in the Literature are not suitable to perform the aforementioned calculation, as they spoil the possibility for the two anyons to contact interact in the most general way. In sect. 3 we present two equivalent procedures (dimensional and ζ- regularization

[15,16]) which, once tested on the standard Hamiltonian in such a way that the known result are recovered, will be applied in sect. 4 to solve our main problem. On the way we shall compute the SVC for contact interacting particles in two and three dimensions.

2. The system of two anyons

The standard classical Hamiltonian for the system of two scalar non relativistic anyons is given by

$$H = \frac{(\mathbf{p}_1 - e\mathbf{A}(\mathbf{x}_1))^2}{2m} + \frac{(\mathbf{p}_2 - e\mathbf{A}(\mathbf{x}_2))^2}{2m} \ , \tag{2.1}$$

where

$$A^a(\mathbf{x}_1) = A^a(-\mathbf{x}_2) = \frac{\alpha}{e}\epsilon^{ab}\frac{(x_1^b - x_2^b)}{|\mathbf{x_1} - \mathbf{x_2}|^2} \tag{2.2}$$

is the statistical potential, corresponding to an Aharonov-Bohm (AB) flux tube centered at each particle position. Without loss of generality we choose $-1 < \alpha < 0$ and we set $m = 1$. After separation of the center of mass (CM) and the relative coordinates, leaving aside the free motion of the CM, we can recast the relative problem in the form

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial \theta} - \alpha\right)^2\right]\psi_{\rm rel}(r,\theta) = E_{rel}\psi_{\rm rel}(r,\theta) \ , \tag{2.3}$$

which looks exactly as the one particle Hamiltonian in a background AB potential. If we assume the usual single valued factorization

$$\psi_{\rm rel}(r,\theta) = \sum_{l=-\infty}^{+\infty} e^{il\theta}\, R_l(r) \, , \tag{2.4}$$

we obtain the radial equation ($k^2 \equiv E_{\rm rel}$ and $\nu \equiv |l+\alpha|$)

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2} + k^2\right] R_l(r) = 0 \ . \tag{2.5}$$

We note that, if we are "building" anyons starting from bosons, only even integer values of l are admitted in eq. (2.4). Needless to say, we could have started from spinless fermions, taking odd integers into account, the whole following analysis being then performed in a quite analogous way (apart from a subtle difference between the two cases which will be considered later on). The general solution of the above equation is

$$R_l(r) = A_l \, J_\nu(kr) \; + \; B_l \, N_\nu(kr) \tag{2.6}$$

where J_ν and N_ν are the Bessel and Neumann functions of order ν respectively.

In order to select a definite solution, we have to choose the boundary conditions for the radial wave functions at the origin in such a way to ensure the self-adjointness of the Hamiltonian quantum operator. It is well known that, in so doing, the coefficients A_l and $B_l \equiv 0$, $l \neq 0, 1$, are uniquely fixed, up to normalization, but for the S-wave in the boson case and the P-wave in the fermion case, for which one-parameter families of self-adjoint extensions indeed exist*. This mathematical freedom is in fact equivalent to the introduction of what is usually called a contact interaction (or "generalized δ-like interaction"). It is easy to see [13] that those kind of δ-like potentials, in two and three dimensions, are not completely well defined and have to be renormalized in order to achieve a non trivial physical meaning. Within this renormalization procedure the contact interaction becomes scale dependent and, in particular, the symmetry breaking parameter can be chosen to be the energy of the unique bound state and/or the resonance's energy [12-14], which generally arises in the presence of arbitrary boundary conditions at the origin.

To be definite, on the one hand we shall refer to *free anyons* when a pure statistical (AB) interaction is present: in this case the coefficient B_0 in eq. (2.6)

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* In the following we shall discuss extensively the boson case; the fermion case can be treated quite analogously *mutatis mutandis*.

is set identically equal to zero even for the S-wave. In so doing the wave function vanishes when two free anyons coincide. On the other hand we shall refer to *contact interacting anyons* when the coefficient B_0 in eq. (2.6) does not vanish: in this case a further "generalized δ-like interaction" is present besides to and independently from the statistical (AB) interaction.

Taking the normalization in the continuum into account (in the tempered distributions topology)

$$\mathcal{S}' - \lim_{R\to\infty} \int_0^R rdr \int_0^{2\pi} d\theta \; \psi_{l_1}^\dagger(k_1 r,\theta)\psi_{l_2}(k_2 r,\theta) = \delta_{l_1,l_2}\,\delta(k_1 - k_2) \;, \qquad (2.7)$$

we are able to write a complete set of orthonormal wave functions in the form

$$\begin{aligned} \psi_l(kr,\theta) &= \sqrt{\frac{k}{2\pi}} e^{il\theta} J_{|l+\alpha|}(kr) \;, \qquad\qquad l \neq 0 \;, \\ \psi_0(kr) &= \sqrt{\frac{(k/2\pi)}{1+\sin 2\pi\mu(k)\cos\pi\alpha}}\,[\sin\pi\mu(k) J_{-\alpha}(kr) + \cos\pi\mu(k) J_\alpha(kr)] \;, \end{aligned} \qquad (2.8)$$

with

$$\tan\pi\mu(k) \equiv \mathrm{sgn}(E_0)\left[\frac{|E_0|}{k^2}\right]^{|\alpha|} \;, \qquad (2.9)$$

where $-\infty \leq E_0 < +\infty$ is the energy parameter which labels the one parameter family of self-adjoint extensions of the radial AB hamiltonian of zero angular momentum, together with the normalized bound state ($E_B \equiv -|E_0|$)

$$B(\kappa_0 r) = (\kappa_0/\pi)\sqrt{\frac{\sin\pi\alpha}{\alpha}}\; K_\alpha(\kappa_0\, r)\,, \qquad\qquad \kappa_0^2 \equiv -E_B\,. \qquad (2.10)$$

As a matter of fact, we recall [11] that in all and only the cases in which $-\infty < E_0 < 0$, there always exists a bound state whose energy is precisely $E_B = E_0$. Furthermore, the case $E_0 \to -\infty$ corresponds to the original AB quantum hamiltonian [10]. For completeness and later use we also quote the phase shifts corresponding to the AB potential and a contact interaction potential: namely,

$$\begin{aligned} \delta_l &= \frac{\pi}{2}|\alpha|\mathrm{sgn}(l) = \frac{\pi}{2}\,[|l| - |l+\alpha|] \;, \qquad l \neq 0 \;, \\ \delta_0(k;\alpha) &= \frac{\pi}{2}\,\alpha - \mathrm{arctg}\,\frac{\sin\pi\alpha}{\cos\pi\alpha + \mathrm{sgn}(E_0)(k^2/|E_0|)^\alpha} \;. \end{aligned} \qquad (2.11)$$

It is worthwhile to stress that, in the absence of the statistical (AB) potential, the last formula reduces to the one appropriate for the purely contact interaction, *viz.*

$$\begin{aligned} \delta_l &= 0\,, \qquad l \neq 0, \\ \delta_0(k) &= \mathrm{arctg}\,\frac{\pi}{\log(-k^2/E_B)}\ . \end{aligned} \tag{2.12}$$

As a new feature we observe that, from eq.s (2.11-12), the existence of resonances is also apparent: their energies read

$$\begin{aligned} E_{\rm res} &= |E_B|(\sec\pi\alpha)^{1/|\alpha|}\ , \qquad \text{if} \qquad 0 < |\alpha| < 1/2\ , \\ E_{\rm res} &= |E_B|\ , \qquad \text{if} \qquad 1/2 < |\alpha| < 1\ , \\ E_{\rm res} &= E_0|\sec\pi\alpha|^{1/|\alpha|}\ , \qquad \text{if} \qquad 1/2 < |\alpha| < 1\,, \quad E_0 > 0. \end{aligned} \tag{2.13}$$

It is easy to verify that all the above expressions for contact interacting particles can be continuosly derived from the ones in presence of an AB potential taking the limit $\alpha \to 0$. This fact is a peculiar characteristic of the scalar boson field and it is interesting to note that it is not maintened in the case of fermions. For the latter one no contact interactions are admissible since the absence of the S-wave in the spectrum reveals that, in this case, the Hamiltonian is essentially self-adjoint. The presence of some AB potential modifies the situation: the P-wave is allowed to fulfil general boundary conditions and, thereby, solutions with square integrable singularities at the origin have to be considered. Besides it is worthwhile to recall that exactly the same features are shared by the relativistic analogue of the two anyons system : a Dirac particle in AB background field. In this case the Svedsen's theorem states that the free Dirac operator is essentially self-adjoint in *2+1* dimensions [17], while the introduction of the AB potential, once again, modifies its nature [18].

To conclude this brief overview on the known results we also consider the system of two anyons in the harmonic well. The introduction of this "confining" potential has been widely used, since the very first studies on the system, to the aim of discretizing the spectrum and to find proper orthonormalizable wavefunctions

[1]. The procedure is well known: first physical quantities are calculated in this "regularized" scheme, then the harmonic coupling is let to zero and the free result regained. As we will see in a moment, this widespread technique turns out to be technically awkward when applied to the Hamiltonian operator of two contact interacting anyons, although quite convenient when applied to free anyons.

The relative Hamiltonian of two anyons in the harmonic well reads

$$H_{\rm rel}\psi_{\rm rel} = \left[(\mathbf{p} - e\mathbf{A}_{rel})^2 + \frac{1}{4}\omega^2 r^2\right]\psi_{\rm rel} = E_{\rm rel}\psi_{\rm rel}\ , \tag{2.14}$$

where $\mathbf{A}_{rel} = (0, \alpha/e\, r)$ while, for the radial problem, we get

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2} - \frac{1}{4}\omega^2 r^2 + k^2\right]\mathcal{R}_l(r) = 0\ , \tag{2.15}$$

Writing the *ansatz* solution in the form

$$\begin{aligned} \mathcal{R}_l(r) &= e^{-\frac{1}{2}y} y^{\frac{1}{2}\nu} w_l(y)\ , \\ y &= \frac{1}{2}\omega r^2\ , \end{aligned} \tag{2.16}$$

we obtain an equation for $w(y)$: namely,

$$y w_l''(y) + (\nu + 1 - y) w_l'(y) + \left[\frac{k^2}{2\omega} - \frac{1}{2}(\nu + 1)\right] w_l(y) = 0 \tag{2.17}$$

which is easily recognized to be the Kummer equation. The general solution is well known to be

$$\begin{aligned} w_l(y) = &A_l\ \Phi\left(\frac{\nu+1}{2} - \frac{k^2}{2\omega}, \nu + 1; y\right) \\ &+ B_l\ \Psi\left(\frac{\nu+1}{2} - \frac{k^2}{2\omega}, \nu + 1; y\right)\ , \end{aligned} \tag{2.18}$$

where Φ is the degenarate confluent hypergeometric function [19] and Ψ is given by

$$\Psi(\alpha, \gamma; y) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha - \gamma + 1)}\Phi(\alpha, \gamma; y) + \frac{\Gamma(\gamma - 1)}{\Gamma(\alpha)} y^{1-\gamma}\Phi(\alpha - \gamma + 1, 2 - \gamma; y)$$

Now, if one asks the usual condition of proper normalizability for the wave function within the range $]0,\infty[$, then the coefficients A_l and B_l are uniquely fixed for any value of $\nu = |l+\alpha|$. For $l \neq 0$ the request of normalizability at the origin asks for $B_l = 0$, while for S-waves, i.e. $l = 0$, the most general case is obtained by setting $A_0 = 0$. Infact while in the former case the spectrum is determined by imposing the truncation of the Hypergeometric function Φ, which entails

$$E_n = \omega(2n+\nu+1)\ , \tag{2.19}$$

in the latter energy levels are obtained by requesting that the set of solutions is complete and orthonormal, i.e.

$$\int_0^\infty \mathcal{R}_l(r)\mathcal{R}_m(r) r\, dr = \delta_{nm}$$

The above condition, when $m \neq n$, is equivalent to the following one:

$$\frac{\omega}{E_n - E_m}\left[\psi\left(\frac{1+\nu}{2} - \frac{E_n}{2\omega}\right) - \psi\left(\frac{1-\nu}{2} - \frac{E_n}{2\omega}\right)\right]^{-\frac{1}{2}} \times$$
$$\left[\psi\left(\frac{1+\nu}{2} - \frac{E_m}{2\omega}\right) - \psi\left(\frac{1-\nu}{2} - \frac{E_m}{2\omega}\right)\right]^{-\frac{1}{2}} \times$$
$$\left[\sqrt{\frac{\Gamma\left(\frac{1-\nu}{2} - \frac{E_n}{2\omega}\right)\Gamma\left(\frac{1+\nu}{2} - \frac{E_m}{2\omega}\right)}{\Gamma\left(\frac{1+\nu}{2} - \frac{E_n}{2\omega}\right)\Gamma\left(\frac{1-\nu}{2} - \frac{E_m}{2\omega}\right)}} - \sqrt{\frac{\Gamma\left(\frac{1+\nu}{2} - \frac{E_n}{2\omega}\right)\Gamma\left(\frac{1-\nu}{2} - \frac{E_m}{2\omega}\right)}{\Gamma\left(\frac{1-\nu}{2} - \frac{E_n}{2\omega}\right)\Gamma\left(\frac{1+\nu}{2} - \frac{E_m}{2\omega}\right)}}\right] = 0\,; \tag{2.20}$$

here the multiplicative factors are kept in order to exhibit the limits $\nu \to 0$ and $\omega \to 0$. The above equation entails that the following real quantity

$$\varrho = \frac{\Gamma(\frac{1}{2} - \frac{E_n}{2\omega} + \frac{\nu}{2})}{\Gamma(\frac{1}{2} - \frac{E_n}{2\omega} - \frac{\nu}{2})} \tag{2.21}$$

must be independent from n . This means that different spectra are obtained for different values of the ϱ. In other words the real value of this ratio actually labels a one parameter family of self adjoint radial Hamiltonians, with purely discrete spectrum. As a matter of fact, the eigenvalues, corresponding to a specific self-adjoint extension, can be determined graphically (or numerically) from the intersection of the right-hand side of eq. (2.21), as a function of the energy, with the parallel to the energy-axis. It is possible to realize that for $\varrho > \varrho_0 = \Gamma(\frac{1+\nu}{2})/\Gamma(\frac{1-\nu}{2})$, there is one negative eigenvalue, otherwise spectrum is non-negative. It is worthwhile to notice that the presence or absence of such negative eigenvalue, precisely corresponds to presence or absence of a bound state when the harmonic well is removed.

The case in which the Aharonov-Bohm is switched off, can be gained from eq. (2.20), taking the limit $\nu \to 0$, where one gets:

$$\varrho = \psi\left(\frac{1}{2} - \frac{E_n}{2\omega}\right) \tag{2.22}$$

which, again, must be independent from n. It is very instructive to realize that, in order to reproduce the pure contact interaction limit, it is always possible to write $\varrho = \ln\frac{|E_B|}{2\omega}$. Then it follows that, just in this limit, ϱ has to be positive, which means in turn that the bound state is always present, as it is well known.

To sum up, we have seen that there is the possibility in principle to regularize the contact interacting case by means of the harmonic well, i.e. keeping arbitrary self-adjoint extensions into account. Unfortunately the lack of the explicit knowledge of the spectrum, prevents us from calculating any physical quantity. Consequently in order to deal with the contact interaction, an alternative regularization method must be employed, as we shall see below.

3. The second virial coefficient for free anyons

We recall that the equation of state for a gas of N identical particles at the equilibrium temperature T, in a two dimensional container of area A, can be written in the form [20]

$$\frac{PA}{Nk_BT} = \frac{P}{\rho k_BT} = \sum_{l=1}^{\infty} a_l(\rho\lambda_T^2)^{l-1}\,. \tag{3.1}$$

where a_l are the virial coefficients and λ_T the thermal wavelength. It is well known that the SVC depends only upon the two-body Hamiltonian and is given by

$$a_2 = a_2^0 - 2\mathrm{Tr}\,[e^{-\beta H} - e^{-\beta H_0}] \tag{3.2}$$

where H_0 is the free Hamiltonian and a_2^0 is the so called exchange contribution. Thanks to Beth and Uhlenbeck (BU) [21], the following expression for a_2 is also available: namely,

$$a_2 - a_2^0 = -2\left[\sum_b e^{-\beta E_b} + \frac{1}{\pi}\sum_l \int_0^\infty \frac{d\delta_l(E)}{dE}\, e^{-\beta E}\, dE\right] \;, \tag{3.3}$$

where δ_l are the phase shifts and E_b the energies of the bound states. We stress that in this formula there is no direct reference to the interaction potential, but only the S-matrix appears through the phase shifts and the possible bound state energies. Furthermore, one has also to assume that the interacting potential is sufficiently well-behaved, in such a way that the RHS of eq. (3.3) does indeed make sense.

Using the result obtained in the previous section we can immediately compute the SVC in presence of the harmonic regulator: it reads

$$\begin{aligned} a_2 - a_2^0 &= -2\sum_{j=0}^{\infty}\Big[(j+1)e^{-\beta(2j+1-\alpha)\omega} + je^{-\beta(2j+1+\alpha)\omega} \\ &\quad - (j+1)e^{-\beta(2j+1)\omega} - je^{-\beta(2j+1)\omega}\Big] \\ &= -\frac{\cosh(\alpha+1)\beta\omega - \cosh\beta\omega}{\sinh^2\beta\omega} \;. \end{aligned} \tag{3.4}$$

Taking the limit $\omega \to 0$ the above expression reduces to

$$a_2 = -\left[\frac{\alpha^2}{2} + \alpha + \frac{1}{4}\right] \;. \tag{3.5}$$

As a first step towards the determination of the SVC in the presence of a contact interaction we develop here a new technique working directly in the continuum [22], *i.e.* on the whole plane.

To this aim it is convenient to rewrite eq. (3.2) in terms of the two-point thermal Green function. Following Comtet et al. [6] we get

$$a_2 = -\frac{1}{4} - \int d^2\mathbf{r}\; (G_{\mathrm{int}}(\beta;\mathbf{r},\mathbf{r}) + G_{\mathrm{int}}(\beta;\mathbf{r},-\mathbf{r}))\,. \tag{3.6}$$

where

$$G_{\rm int} = G - G_0 \tag{3.7}$$

and

$$\begin{aligned} G_0(\beta;\mathbf{r},\mathbf{r}') &= \left\langle \mathbf{r}|e^{-\beta H_0}|\mathbf{r}'\right\rangle \\ &= \frac{1}{4\pi\beta}\exp\left\{-\frac{(\mathbf{r}-\mathbf{r}')^2}{4\beta}\right\} \end{aligned} \tag{3.8}$$

Now, if we consider the spectral decomposition of the relative Hamiltonian

$$H_{\rm rel} = \sum_{j=-\infty}^{+\infty} e^{2ij(\theta-\theta')} \int_0^\infty \frac{kdk}{2\pi}\ k^2 J_{|2j+\alpha|}(kr) J_{|2j+\alpha|}(kr')\ , \tag{3.9}$$

we have

$$G(\beta;\mathbf{r},\mathbf{r}') = \left\langle \mathbf{r}|e^{-\beta H}|\mathbf{r}'\right\rangle = \frac{1}{2\pi}\int_0^\infty kdk\ \exp\{-\beta k^2\}\times$$
$$\left[\sum_{j=1}^{\infty} e^{2ij(\theta-\theta')} J_{2j+\alpha}(kr) J_{2j+\alpha}(kr') + \sum_{j=0}^{\infty} e^{2ij(\theta-\theta')} J_{2j-\alpha}(kr) J_{2j-\alpha}(kr')\right]\ . \tag{3.10}$$

From eq. (3.6) it is clear that only the contributions coming from coincident and opposite points with respect to the origin have to be considered, *i.e.*

$$\begin{aligned} G(\beta;\mathbf{r},\mathbf{r}) = G(\beta;\mathbf{r},-\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty kdk\ e^{-\beta k^2}\times \\ \left[\sum_{j=0}^{\infty} J^2_{2j+\alpha}(kr) + \sum_{j=0}^{\infty} J^2_{2j-\alpha}(kr) - J^2_\alpha(kr)\right]. \end{aligned} \tag{3.11}$$

So we have to evaluate integrals of the type

$$I_\alpha = \int d^2\mathbf{r} \int_0^\infty dk\, k e^{-\beta k^2} \sum_{j=0}^{\infty} \left[J_{2j+\alpha}(kr)\right]^2\ . \tag{3.12}$$

which is manifestly divergent as it stands. Actually the same kind of problem occurs in the aforementioned calculation of the SVC performed by Arovas et al. [2] and by Comtet et al. [6]. Here we adopt the following definition

$$\begin{aligned} I_\alpha &= \sum_{j=0}^{\infty}\int_0^\infty dk\, k e^{-\beta k^2} \int d^2\mathbf{r}\ \left[J_{2j+\alpha}(kr)\right]^2 \\ &\equiv \lim_{\omega\to 1}\sum_{j=0}^{\infty}\int_0^\infty dk\, k e^{-\beta k^2} \int d^{2\omega}\mathbf{r}\ \left[J_{2j+\alpha}(kr)\right]^2\ . \end{aligned} \tag{3.13}$$

i.e. we employ dimensional regularization and we have explicitely put forward the fact that we are dealing here with boson-made anyons $l = 2j$. This method is standard within perturbative quantum field theory; we have to find a region in the complex ω-plane where the series is well defined and continue analytically the result to $\omega = 1$. The calculation is straightforward, passing to polar coordinates and using

$$\int_0^\infty dr\ r^{2\omega-1}\left[J_{2j+\alpha}(kr)\right]^2 = \frac{k^{-2\omega}}{2\sqrt{\pi}}\frac{\Gamma(2j+\alpha+\omega)}{\Gamma(2j+\alpha+1-\omega)}\frac{\Gamma(\frac{1}{2}-\omega)}{\Gamma(1-\omega)} \equiv C_\alpha^j(\omega)\ k^{-2\omega} \tag{3.14}$$

we get

$$I_\alpha = \lim_{\omega\to 1}\sum_{j=0}^{\infty} C_\alpha^j(\omega)\frac{\Gamma(1-\omega)}{2\beta^{1-\omega}}\ . \tag{3.15}$$

We note that the two divergent terms in eq. (3.15) cancel each other, showing the fact that the thermal two-point Green's function is a well defined quantity in the thermodynamic limit, although a suitable intermediate regularization has to be introduced to give a precise meaning to eq. (3.11). Taking the harmless limit $\omega \to 1$ in eq. (3.15) we get

$$I_\alpha = -\pi\lim_{\omega\to 1}\sum_{j=0}^{\infty}\frac{\Gamma(2j+\alpha+\omega)}{\Gamma(2j+\alpha+1-\omega)}\ . \tag{3.16}$$

As it can be easily seen, this series is convergent for $\omega < 0$ and can be summed exactly, *viz.*

$$\sum_{j=0}^{\infty}\frac{\Gamma(2j+\alpha+\omega)}{\Gamma(2j+\alpha+1-\omega)} = \sum_{j=0}^{\infty} 2^{2\omega-1}\frac{\Gamma\left(j+\frac{\alpha+\omega+1}{2}\right)}{\Gamma\left(j+\frac{\alpha+1-\omega}{2}\right)} \times\left[1+\frac{\Gamma\left(j+\frac{\alpha+\omega}{2}\right)-\Gamma\left(j+1+\frac{\alpha-\omega}{2}\right)}{\Gamma\left(j+1+\frac{\alpha-\omega}{2}\right)}\right] = 2^{2\omega-1}F\left(1,\frac{\alpha+\omega+1}{2};\frac{\alpha+1-\omega}{2};1\right)\frac{\Gamma\left(\frac{\alpha+\omega+1}{2}\right)}{\Gamma\left(\frac{\alpha+1-\omega}{2}\right)}+\mathcal{R}_\alpha(\omega)\ , \tag{3.17}$$

where $F(a,b;c;z)$ is the hypergeometric function, while evidently $\mathcal{R}_\alpha(\omega) \to 0$ when $\omega \to 1$, owing to

$$\mathcal{R}_\alpha(\omega) = \frac{2^{2\omega-1}\Gamma\left(\frac{\alpha+\omega+1}{2}\right)}{\Gamma\left(\frac{\alpha+1-\omega}{2}\right)\Gamma\left(1+\frac{\alpha-\omega}{2}\right)} \times$$
$$\left\{\Gamma\left(\frac{\alpha+\omega}{2}\right)\ {}_3F_2\left(1,\frac{\alpha+\omega+1}{2},\frac{\alpha+\omega}{2};\frac{\alpha-\omega+1}{2},1+\frac{\alpha+\omega}{2};1\right)\right.$$
$$\left. -\,\Gamma\left(1+\frac{\alpha-\omega}{2}\right)\ {}_3F_2\left(1,1+\frac{\alpha-\omega}{2},\frac{\alpha+\omega+1}{2};\frac{\alpha-\omega+1}{2},1+\frac{\alpha+\omega}{2};1\right)\right\}\ .$$

From the well known formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3.18}$$

we finally get :

$$I_\alpha = \lim_{\omega\to 1} \pi \frac{2^{2\omega-1}}{1+\omega}\frac{\Gamma\left(\frac{\alpha+\omega+1}{2}\right)}{\Gamma\left(\frac{\alpha-\omega-1}{2}\right)} = \frac{\pi}{2}\alpha(\frac{\alpha}{2}-1). \tag{3.19}$$

Collecting now all the terms in eq. (3.11) we finally obtain the desired result as in eq. (3.5). It is worthwhile to stress once again that, in general, the choice of the regularization that is needed to perform the calculations is, of course, just a matter of convenience. Nevertheless, as the forthcoming analysis will show, the method we have here presented turns out to be more powerful, since it can be applied to the most general case, at variance with the other ones available in the Literature.

Before ending this section it is quite natural to ask whether the same type of regularization can be applied to the BU formula. As it has been already noticed [6], a careless insertion of (2.11) into (3.3) leads to the wrong result of a null contribution to the SVC (actually, in the absence of the contact interaction, the phase shifts are energy independent). On the other hand, it has been suggested [6,23] that the density of states in the AB potential is all concentrated at zero energy. This latter statement exactly corresponds to assume for the phase shifts the following form: namely,

$$
\begin{aligned}
\delta_l(E) &= \frac{\pi}{2}\left[|l| - |l+\alpha|\right]\vartheta(E) \qquad (l \neq 0) \\
\delta_0(E;\alpha,E_0) &= \left[\frac{\pi}{2}\,\alpha + \mathrm{sgn}(E_0)\mathrm{arctg}\,\frac{\sin\pi\alpha}{\cos\pi\alpha + \mathrm{sgn}(E_0)(E/|E_0|)^{\alpha}}\right]\vartheta(E)
\end{aligned}
\tag{3.20}
$$

where $\vartheta(E)$ is the Heaviside's step distribution. Following this idea we can easily obtain the formal expressions

$$
\begin{aligned}
a_2^{\pm} - a_2^0 &= -2\frac{1}{\pi}\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty}\frac{\pi}{2}\left[|2j| - |2j\pm\alpha|\right]\delta(E)e^{-\beta E}\,dE \\
&= -\sum_{j=-\infty}^{+\infty}\left[|2j| - |2j\pm\alpha|\right] ,
\end{aligned}
\tag{3.21}
$$

where the plus sign corresponds to free anyons, namely $E_0 \to -\infty$, whereas the minus sign describes the special case of "maximally repulsive" contact interaction, in which the wave function is purely singular, namely $E_0 = 0$. In both cases (which have also been discussed in [24]) the bound states are not present although a phase shift flip occurs.

We notice that the result of eq. (3.5) is immediately obtained once the ζ-regularization is applied to the sum in eq. (3.21) with the plus sign: namely,

$$
\sum_{j=0}^{\infty}(2j\pm\alpha) \equiv 2\zeta\left(-1,\frac{1+\alpha}{2}\right) - \zeta(-1,0) = -\frac{1}{2}\left(\frac{\alpha^2}{2}\mp\alpha\right) . \tag{3.22}
$$

where $\zeta(s,q)$ is the Riemann-Hurwitz ζ-function. Furthermore the result with the minus sign is obtained as well and corresponds to the replacement $\alpha \to -\alpha$.

It should be gathered that, within this framework, the derivative in the integrand of eq. (3.3) has to be understood in the sense of the distributions. As we will prove in the next section, both the above described procedures, *i.e.* dimensional regularization applied to eq. (3.6) and ζ-regularization to the BU formula (3.3), give the same result for the contact interacting system too. The only difference is that the former method does not require any *ad hoc* hypothesis and, in this sense, it can be considered to implement *a fortiori* the introduction of the step distribution in eq. (2.11).

4. The virial coefficient for contact interacting particles

The analyis performed in the previous section proved the direct equivalence between the phase shifts approach and the usual thermal Green's function method in the evaluation of the SVC for the pure AB interaction. It has been shown that the long range nature of the AB potential can be handled with appropriate dimensional and analytic regularization procedures in the continuum. Thanks to these preliminary results it is now clear that the introduction of the contact interaction does not require any further particular care, provided the above mentioned regularizations are fully employed. At this point, all we have to do is simply to separate the two contributions, the contact interaction from the statistical (AB) potential. For the S-wave contribution to the SVC we have

$$I_\alpha^0 = \lim_{\omega\to 1} \frac{2\pi^\omega}{\Gamma(\omega)} \int_0^\infty dk \frac{k e^{-\beta k^2}}{1+\mathrm{tg}^2\,\pi\mu(k) + 2\,\mathrm{tg}\,\pi\mu(k)\cos\alpha\pi} \times \int_0^\infty dr r^{2\omega-1} \left[\mathrm{tg}^2\pi\mu(k)\, J_{-\alpha}^2(kr) + J_\alpha^2(kr) + 2\,\mathrm{tg}\,\pi\mu(k)\, J_{-\alpha}(kr)\, J_\alpha(kr)\right] . \tag{4.1}$$

Using eq. (3.14) and the equality

$$\int_0^\infty dr\; r^{2\omega-1}\; J_\alpha(kr)\, J_{-\alpha}(kr) = \frac{k^{-2\omega}}{2\sqrt{\pi}} \frac{\Gamma(1/2-\omega)\Gamma(\omega)}{\Gamma(1-\omega-\alpha)\Gamma(1-\omega+\alpha)} \tag{4.2}$$

we get

$$I_\alpha^0 = \lim_{\omega\to 1} \frac{\pi^{\omega-1/2}}{2^\omega\Gamma(\omega)} \int_0^\infty dE \frac{\Gamma(1/2-\omega)|E_0|^{|\alpha|}E^{\alpha-\omega}e^{-\beta E}}{1+(|E_0|/E)^{2|\alpha|}+2\mathrm{sgn}(E_0)\,(|E_0|/E)^{|\alpha|}\cos\pi\alpha} \times \left\{ \frac{\pi(|E_0|/E)^{-|\alpha|}\csc\pi(\omega+\alpha) + \pi(|E_0|/E)^{|\alpha|}\csc\pi(\omega-\alpha) + 2\mathrm{sgn}(E_0)\Gamma(1-\omega)}{\Gamma(1-\omega)\Gamma(1-\omega-\alpha)\Gamma(1-\omega+\alpha)} \right\} . \tag{4.3}$$

We note that the above expression does reproduce the known results in the limits $E_0 \to -\infty$ (regular S-wave function) and $E_0 \to 0$ (purely singular S-wave function) in which bound states are absent: namely,

$$a_2(\alpha, E_0 \to -\infty, T) = a_2(\alpha) \ , \qquad a_2(\alpha, E_0 = 0, T) = a_2(-\alpha) \, . \tag{4.4}$$

If we now assume E_0 to be finite and different from zero, we can change the integration variable $E \mapsto x \equiv E/|E_0|$ and rewrite eq. (4.3) in the form

$$\begin{aligned} I^0_\alpha = \lim_{\omega\to 1} \frac{\pi^{\omega-1/2}\Gamma(1/2-\omega)}{2^\omega\Gamma(\omega)\Gamma(1-\omega)} \int_0^\infty dx \frac{x^{\alpha-\omega}|E_0|^{1-\omega}\ e^{-\beta|E_0|x}}{1+2\mathrm{sgn}(E_0)x^\alpha\cos\pi\alpha + x^{2\alpha}} \\ \times\left\{\frac{\pi x^\alpha \csc\pi(\omega-\alpha) + \pi x^{-\alpha}\csc\pi(\omega+\alpha) + 2\mathrm{sgn}(E_0)\Gamma(1-\omega)}{\Gamma(1-\omega-\alpha)\Gamma(1-\omega+\alpha)}\right\} \ . \end{aligned} \tag{4.5}$$

The limit $\omega \to 1$ in the first integral above is immediately performed once we recall that

$$\mathcal{S}' - \lim_{\epsilon\to 0} \epsilon|x|^{\epsilon-1} = 2\delta(x) \ ; \tag{4.6}$$

in so doing we eventually remain with the simple general expression for the SVC, in terms of the following integral representation: namely,

$$\begin{aligned} a_2(\alpha, E_0, T) = a_2(\alpha) - 2\theta(-E_0)e^{-\beta E_0} \\ - \frac{\alpha\sin\pi\alpha}{\pi}\int_0^\infty \frac{dx}{x^{1+\alpha}} \frac{2\mathrm{sgn}(E_0)\ e^{-\beta|E_0|x}}{1+2\mathrm{sgn}(E_0)x^{|\alpha|}\cos\pi\alpha + x^{2|\alpha|}} \ , \end{aligned} \tag{4.7}$$

where the last integral is manifestly convergent. Before analyzing some special cases, we note that, as it can be easily verified, if the phase shifts of eq. (3.23) are introduced into (3.3), exactly the same result is regained.

First of all it is interesting to look in the high temperature limit ($kT >> |E_0|$), of the above expression, *i.e.* in the domain in which the virial expansion is meaningful. Using the integral

$$\int_0^\infty dx \frac{x^{-1-\alpha}}{1+2\mathrm{sgn}(E_0)\mathrm{cos}\pi\alpha\, x^{|\alpha|} + x^{2|\alpha|}} = \frac{\pi}{\sin(\pi\alpha)} \ , \tag{4.8}$$

we get, up to the zeroth-order in β,

$$a_2(\alpha, E_0, T) \to a_2(\alpha) - 2 - \frac{2}{\pi}\alpha \ . \tag{4.9}$$

In the absence of the AB potential we obtain the SVC for contact interacting particles in two dimensions [25]

$$
\begin{aligned}
a_2(0, E_B, T) &= a_2^0 - 2\Big\{ e^{\beta|E_B|} - \int_0^\infty \frac{dE}{E} \frac{e^{-\beta E}}{\log^2(-E/E_B) + \pi^2} \Big\} \\
&= a_2^0 - 2\nu(\beta|E_B|)
\end{aligned}
\tag{4.10}
$$

where $-\infty \leq E_B < 0$ and

$$
\nu(x) \equiv \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt \ . \tag{4.11}
$$

For $\alpha = -(1/2)$ the integral in eq. (4.7) reduces to [19]

$$
\int_0^\infty \frac{dx}{\sqrt{x}} \frac{e^{-\beta|E_0|x}}{1+x} = \pi\, e^{\beta|E_0|} \operatorname{erfc}(\sqrt{\beta|E_0|}) \ , \tag{4.12}
$$

which is related to the SVC for contact interacting particles in three dimensions. As a matter of fact we get

$$
a_2^{3\mathrm{D}}(E_0, T) + \frac{1}{4\sqrt{2}} = \sqrt{2}\left[a_2^{2\mathrm{D}}(-\frac{1}{2}, E_0, T) + \frac{1}{4} \right] \ . \tag{4.13}
$$

In fact this curious dimensional transmutation phenomenon has been already noticed by Manuel and Tarrach [11] for the phase shifts. Its origin can be traced back to the parametrization of the radial equation [12].

5. Conclusions

We have calculated the second virial coefficient for a gas of contact interacting anyons. The difficulties within the previously available techniques have been overcome with the introduction of new regularization procedures which work directly in the continuum. The ζ-function technique allowed us to answer adfirmatevely to the open question concerning the possibility of using the BU formula in the presence of long-range potentials as the AB one. Finally, as particular cases, the SVC for contact interacting particles in two and three dimensions were obtained.

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