EXACT SOLUTION OF THE 1-IMPURITY QUANTUM HALL PROBLEM

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0. INTRODUCTION

- EXPERIMENTAL set of the Integer Quantum Hall Effect

  `Klaus von Klitzing (1980)`

  ◦ QUANTIZED PLATEAUX IN $R_H = \sigma_{xy}^{-1}$

  ◦ VANISHING σ_{xx} IN QUANTUM HALL REGIME

- THEORETICAL "*mesoscopic*" explanation

  ◦ IDEAL 2D ELECTRON GAS

  electron interactions are negligible

  ◦ GAUGE INVARIANCE (`Laughlin 1984`)

  to explain R_H quantization

  ◦ DISORDER/IMPURITIES (`Prange 1981`)

  to explain plateaux

- the 1-particle quantum mechanical problem to be solved

$$H = \frac{1}{2m}\left\{\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})\right\}^2 - \frac{e}{mc}\mathbf{s}\cdot\mathbf{B} - e\mathbf{x}\cdot\mathbf{E}_H + V_{\rm dis}$$

- $\mathbf{s} = \frac{\hbar}{2}\vec{\sigma}$ is the electron spin operator

- $B > 0$ is the uniform magnetic field along the Ox_3 axis

- gauge potential in the symmetric gauge

$$A_j(x_1, x_2) = -\epsilon_{jl}x_l\frac{B}{2}\ , \quad j, l = 1, 2\ , \quad \epsilon_{12} = 1\ ,$$

- the Hall field $\mathbf{E}_H$ is uniform to a high degree of accuracy

- $V_{\rm dis}$ is the potential describing DISORDER/IMPURITIES

- Consider the PURE 2D IDEAL electron gas $V_{\text{dis}} = 0$

○ ALL the electrons lie in the HALL CONDUCTING LANDAU BANDS

○ EACH STATE carries the Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{h}$$

○ the TOTAL Hall conductance is

$$\Sigma_{xy} = -\frac{e^2}{h}\nu$$

$$\nu \equiv \frac{\langle \mathtt{n} \rangle}{\Gamma_L}$$ Landau band filling

$$\Gamma_L = \frac{eB}{hc}$$ Landau degeneracy

$\langle \mathtt{n} \rangle$= average # of electrons$\times cm^{-2}$

THE P-J MECHANISM

`Prange (1981) Joynt-Prange (1982)`

- Consider the IMPURE 2D IDEAL electron gas $\quad V_{\mathrm{dis}} \neq 0$

○ there exist BOUND STATES which
DO NOT CARRY HALL CURRENT

$$\Updownarrow$$

○ the filling of Landau bands DECREASES

$$\nu \quad \longmapsto \quad \frac{\nu}{1+\epsilon} \qquad \epsilon > 0$$

○ the electron Hall conductivity just INCREASES

$$\sigma_{xy} = -\frac{e^2}{h}(1+\epsilon)$$

- the net result of the P-J mechanism is therefore

○ THE HALL CONDUCTANCE IS AGAIN

$$\Sigma_{xy} = -\frac{e^2}{h}\nu$$

○ CHANGE IN DENSITY $\Leftrightarrow$ FERMI ENERGY

$$2mE_F \simeq \left(h^3 \langle \mathtt{n} \rangle\right)^{2/3}$$

BETWEEN NEXT-TO-LYING LANDAU BANDS

$$\Updownarrow$$

○ NO CHANGE IN THE HALL CONDUCTANCE Σ_{xy}

$$\Updownarrow$$

○ **HALL PLATEAUX**

: QUESTION :

¿ IS IT POSSIBLE TO CHECK

THE P-J MECHANISM

IN SOME EXACTLY SOLVABLE MODEL ?

`Prange (1981)` SUGGESTED IT IS POSSIBLE

IF WE MODEL DISORDER IN TERMS OF

$V_{\rm dis}$ WITH A **POINT-LIKE** SUPPORT

NOTATIONS

- holomorphic dimensionless coordinates

$$z = \frac{x_1 + i x_2}{\lambda_B} = x + iy$$

$$x_1 = \lambda_B \frac{z + \bar{z}}{2} \ , \ x_2 = \lambda_B \frac{z - \bar{z}}{2i}$$

$$\partial_z = (\lambda_B/2)\,(\partial_1 - i\partial_2)$$

- $\lambda_B = \sqrt{\hbar c/eB}$ magnetic length
- $\varrho \equiv 2(E_H/B)\sqrt{mc^2/\hbar\omega}$ Hall parameter
- $\omega \equiv (eB/mc)$ cyclotron's angular frequency

1. SOLVED MODELS

- The PURE Landau model $V_{\text{dis}} = 0$ `Landau (1930)`

ENERGY creation-distruction operators

$$\delta_\varrho \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z-\varrho}{4}\right\} = \bar{\delta}_\varrho^\dagger$$

$$\bar{\delta}_\varrho \equiv i\sqrt{2}\left\{\partial_z - \frac{\bar{z}-\varrho}{4}\right\} = \delta_\varrho^\dagger$$

DEGENERACY creation-distruction operators

$$\theta_\varrho \equiv -i\sqrt{2}\left\{\partial_z + \frac{\bar{z}-\varrho}{4}\right\} = \bar{\theta}_\varrho^\dagger$$

$$\bar{\theta}_\varrho \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z-\varrho}{4}\right\} = \theta_\varrho^\dagger$$

- OPERATOR ALGEBRA

$$\left[\delta_\varrho, \bar{\delta}_\varrho\right] = \left[\theta_\varrho, \bar{\theta}_\varrho\right] = 1 \ , \quad [\delta_\varrho, \theta_\varrho] = \left[\delta_\varrho, \bar{\theta}_\varrho\right] = 0$$

- SPIN-UP (↑) SCHRÖDINGER-PAULI HAMILTONIAN

$$H_\uparrow(E_H) \equiv \frac{\hbar^2}{2m\lambda_B^2}\mathtt{h}(\varrho)$$

◦ RESCALED $\mathtt{h}$AMILTONIAN in units of energy $(\hbar^2/2m\lambda_B^2)$

$$\mathtt{h}(\varrho) = 2\bar{\delta}_\varrho\delta_\varrho + i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2$$

◦ DEGENERACY SPLITTING OPERATOR

$$\mathtt{T}(\varrho) \equiv i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2$$

CONTINUOUS SPECTRUM & EIGENSTATES

$$\varepsilon_{n,p_\perp} = 2n - \varrho p_\perp - \frac{1}{4}\varrho^2$$

$$n+1 \in \mathbf{N} \qquad p_\perp = \frac{p_2 \lambda_B}{\hbar} \in \mathbf{R}$$

- IMPROPER NON-DEGENERATE eigenfunctions

$$\psi_{n,p_\perp}(x,y;\varrho) = u_n\left(x - \frac{1}{2}\varrho - p_\perp\right)\exp\left\{iyp_\perp - i\frac{xy}{2}\right\}\sqrt{\frac{1}{2\pi}}$$

$$\{u_k \mid k+1 \in \mathbf{N}\} \quad \text{HERMITE's functions basis}$$

- ORTHONORMALITY and COMPLETENESS

$$\langle \psi_{n,p_\perp}(\varrho) | \psi_{m,q_\perp}(\varrho) \rangle = \delta_{n,m}\ \delta(p_\perp - q_\perp)$$

$$p_\perp, q_\perp \in \mathbf{R} \qquad n+1, m+1 \in \mathbf{N}$$

- The Hall CURRENT OPERATOR is

$$J_H = \frac{ie\hbar}{m\lambda_B\sqrt{2}}\left(\delta_\varrho - \bar{\delta}_\varrho + \frac{i}{\sqrt{2}}\varrho\right)$$

EACH NON-DEGENERATE STATE $|\psi_{n,p_\perp}\rangle$

CARRIES THE SAME HALL CURRENT

$$\langle\psi_{n,p_\perp}|\, J_H \,|\psi_{n,p_\perp}\rangle = \sigma_{xy}\ E_H$$

$$\Updownarrow$$

- the 1-ELECTRON STATE Hall conductivity is

$$\sigma_{xy} = -\frac{e^2}{h}\Gamma_L^{-1}$$

$V_{\text{dis}} \neq 0$ 1-IMPURITY MODEL $\mathbf{E}_H = 0$

1-IMPURITY described by

POINT-LIKE MAGNETIC VORTEX

`Y. Aharonov, D. Bohm (1959)`

`A. Comtet, Y. Georgelin, S. Ouvry (1989)`

`J. Desbois, S. Ouvry, C. Texier (1997)`

$$A_j(x_1, x_2) = -\epsilon_{jl} x_l \left[\frac{B}{2} - \frac{(\phi/2\pi)}{x_1^2 + x_2^2} \right]$$

- ◦ QUANTUM FLUX UNITY $\phi_0 \equiv (hc/e)$

- ◦ FLUX PARAMETER $\alpha \equiv (\phi/\phi_0)$

- ◦ INTEGER α is NOT OBSERVABLE $\Rightarrow -1 < \alpha \leq 0$

- SINGULAR creation-distruction operators $\qquad \gamma \equiv (\bar{z}z/2)$

$$\delta(\alpha) \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\delta}^{\dagger}(\alpha)$$

$$\bar{\delta}(\alpha) \equiv i\sqrt{2}\left\{\partial_{z} - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \delta^{\dagger}(\alpha)$$

$$\theta(\alpha) \equiv -i\sqrt{2}\left\{\partial_{z} + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\theta}^{\dagger}(\alpha)$$

$$\bar{\theta}(\alpha) \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \theta^{\dagger}(\alpha)$$

$(1/[\gamma])$-SINGULARITY IS DEFINED
ACCORDING TO THE
THEORY OF DISTRIBUTIONS

$$\frac{1}{[\gamma]} = \frac{1}{4}\triangle \ln^{2}(\bar{z}z) + C\delta^{(2)}(\bar{z}, z)$$

C ARBITRARY CONSTANT

- DOMAIN of the SINGULAR operators

$$\mathcal{T}(\mathbf{R}^2) = \{f \in \mathcal{S}(\mathbf{R}^2)|\ f(0) = 0\}$$

dense in $L^2(\mathbf{R}^2)$

- SINGULAR OPERATORS ALGEBRA

$$[\delta(\alpha), \bar{\delta}(\alpha)] = [\theta(\alpha), \bar{\theta}(\alpha)] = 1$$

$$[\delta(\alpha), \theta(\alpha)] = [\delta(\alpha), \bar{\theta}(\alpha)] = 0$$

- ANGULAR MOMENTUM & $\mathtt{h}$AMILTONIAN

$$\mathtt{L} = \hbar\{|\alpha|\mathbf{1} + \bar{\theta}(\alpha)\theta(\alpha) - \bar{\delta}(\alpha)\delta(\alpha)\}$$

$$\mathtt{h}(\alpha) = 2\bar{\delta}(\alpha)\delta(\alpha)$$

$$[\mathtt{h}(\alpha), \mathtt{L}] = 0$$

DISCRETE SPECTRUM & EIGENSTATES

- INTEGER valued ∞-**DEGENERATE** LEVELS

$$\check{\varepsilon}_n = 2n \qquad \mathtt{L} > 0$$

- ◦ **REGULAR** EXTENDED LANDAU BANDS

$$\Psi_{n<k}(z,\bar{z}) = (-1)^n \sqrt{\frac{n!}{2\pi\Gamma(k+\alpha+1)}} \left(i\frac{z}{\sqrt{2}}\right)^{k-n}$$

$$\times\ \gamma^{-|\alpha|/2} \exp\{-\gamma/2\} L_n^{(k-n+\alpha)}(\gamma)$$

$$k \geq n+1 \in \mathbf{N} \qquad \ell = k-n > 0$$

$L_n^{(\beta)}$ Laguerre polynomials

- REAL valued $(\mathbf{n+1})$-**DEGENERATE** LEVELS

$$\hat{\varepsilon}_n = 2(n-\alpha) \qquad \mathtt{L} \leq 0$$

- ◦ **REGULAR** LOCALIZED EIGENSTATES

$$\Psi_{n\geq k}(z,\bar{z}) = (-1)^k \sqrt{\frac{k!}{2\pi\Gamma(n-\alpha+1)}} \left(-i\frac{\bar{z}}{\sqrt{2}}\right)^{n-k}$$

$$\times\ \gamma^{|\alpha|/2} \exp\{-\gamma/2\} L_k^{(n-k-\alpha)}(\gamma)$$

$$n+1 \in \mathbf{N} \qquad 0 \leq k \leq n \qquad \ell = k-n \leq 0$$

WITHOUT HALL FIELD $\mathbf{E}_H = 0$

THE WHOLE SET OF EIGENSTATES

IS ORGANIZED ACCORDING TO THE P-J PICTURE

: QUESTION :

¿ IS THIS **REGULAR** SOLUTION
THE **ONLY ONE** ALLOWED BY
GENERAL PRINCIPLES OF QM ?

THE ANSWER IS **NO**

2. SELF-ADJOINT EXTENSIONS $\mathbf{E}_H = 0$

- RADIAL HAMILTONIANS ARE ONLY **SYMMETRIC**

- THEOREM :

IN EACH SUBSPACE OF FIXED

INTEGER ANGULAR MOMENTUM $\ell \in \mathbf{Z}$

THE SYMMETRIC RADIAL HAMILTONIANS

ADMIT A 1-PARAMETER FAMILY OF

SELF-ADJOINT EXTENSIONS (SAE)

- DIFFERENT SAE HAVE DIFFERENT SPECTRA

- SAE **DOMAINS** CONTAIN L^2-SINGULARITIES

CONTACT-INTERACTION

EXAMPLE : $\ell = 0$ $\alpha \neq 0$

- **SINGULAR** COMPLETE O.N. SET $\in L^2(\mathbf{R}^2)$

$$\Phi_{n,n}(z,\bar{z}) = (-1)^n \sqrt{\frac{n!}{2\pi\Gamma(n+\alpha+1)}}$$

$$\times\gamma^{-|\alpha|/2}\exp\{-\gamma/2\}L_n^{(\alpha)}(\gamma) \qquad n+1 \in \mathbf{N}$$

- RIESZ-FISHER EXPANSION ON THE REGULAR SET

$$\Phi_{n,n}(z,\bar{z}) = \sum_{j=0}^{\infty} \check{c}_{n,0}^{\,j}\Psi_{j,j}(z,\bar{z})$$

$$\check{c}_{n,0}^{\,j} \equiv \binom{j-\alpha}{n}\binom{n+\alpha}{j}\sqrt{\frac{j!n!}{\Gamma(j-\alpha+1)\Gamma(n+\alpha+1)}}$$

$$\Psi_{j,j}(z,\bar{z}) = (-1)^j \sqrt{\frac{j!}{2\pi\Gamma(j-\alpha+1)}}$$

$$\times\gamma^{|\alpha|/2}\exp\{-\gamma/2\}L_j^{(-\alpha)}(\gamma) \qquad j+1 \in \mathbf{N}$$

SPECTRAL DECOMPOSITIONS $\quad \ell = 0$

- REGULAR SAE :

$$\mathtt{h}_0(\alpha) = \sum_{n=0}^{\infty} 2(n + |\alpha|) \, |\Psi_{n,n}\rangle \, \langle \Psi_{n,n}|$$

- SINGULAR SAE :

$$\mathtt{H}_0(\alpha) = \sum_{n=0}^{\infty} 2n \, |\Phi_{n,n}\rangle \, \langle \Phi_{n,n}|$$

THEY CORRESPOND TO THE TWO

LINEARLY INDEPENDENT SOLUTIONS OF

THE 2^{nd} ORDER DIFFERENTIAL OPERATOR

OF THE $\quad \ell = 0 \quad$ RADIAL HAMILTONIAN

INTERPOLATING $\Rightarrow$ 1-PARAMETER FAMILY

REMARKS

- THE SAME CONSTRUCTION $\forall \ell \in \mathbf{Z}$

- RESULT : ∞-SET OF SAE

- SELECTED SAE ITEM :

$$\hat{\Delta}(\alpha) \equiv \sum_{n=0}^{\infty} 2(n+|\alpha|) \times$$

$$\left\{ \sum_{k=0}^{n} |\Psi_{n\geq k}\rangle \langle \Psi_{n\geq k}| + \sum_{k=n+1}^{\infty} |\Phi_{n<k}\rangle \langle \Phi_{n<k}| \right\}$$

◦ DISCRETE REAL-VALUED SPECTRUM

◦ ∞-DEGENERATE ENERGY LEVELS

◦ REGULAR ($\ell \leq 0$) & SINGULAR ($\ell > 0$) EIGENSTATES

$\hat{\Delta}(\alpha)$ **SAE** $\qquad \hat{\varepsilon}_n = 2(n + |\alpha|)$

7	$\otimes$	$\otimes$	$\otimes$	$\otimes$
6	$\otimes$	$\otimes$	$\otimes$	$\otimes$
5	$\otimes$	$\otimes$	$\otimes$	$\otimes$
4	$\otimes$	$\otimes$	$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$	$\odot$
2	$\otimes$	$\otimes$	$\odot$	$\odot$
1	$\otimes$	$\odot$	$\odot$	$\odot$
0	$\odot$	$\odot$	$\odot$	$\odot$
$k^{\uparrow}/\hat{\varepsilon}_{\rightarrow}$	$2\|\alpha\|$	$2+2\|\alpha\|$	$4+2\|\alpha\|$	$6+2\|\alpha\|$

3. 1-IMPURITY EXACT SOLUTION

- the rescaled $\mathtt{h}$AMILTONIAN

$$\frac{2m\lambda_B^2}{\hbar^2} H(\alpha, E_H) \equiv \mathtt{h}(\alpha, \varrho) + \mathtt{T}(\alpha, \varrho)$$

$$\mathtt{h}(\alpha, \varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$$

- DEGENERACY SPLITTING operator

$$\mathtt{T}(\alpha, \varrho) \equiv i\frac{\varrho}{\sqrt{2}}\left[\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)\right] - \frac{3}{4}\varrho^2$$

CREATION-DESTRUCTION ALGEBRA

$$\delta_\varrho(\alpha) \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\}$$

$$\bar{\delta}_\varrho(\alpha) \equiv i\sqrt{2}\left\{\partial_{z} - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\}$$

$$\theta_\varrho(\alpha) \equiv -i\sqrt{2}\left\{\partial_{z} + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\}$$

$$\bar{\theta}_\varrho(\alpha) \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\}$$

$$[\delta_\varrho(\alpha), \bar{\delta}_\varrho(\alpha)] = [\theta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 1$$

$$[\delta_\varrho(\alpha), \theta_\varrho(\alpha)] = [\delta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 0$$

• COMMUTATION RELATIONS •

$$\mathtt{H}(\alpha,\varrho) \equiv \mathtt{h}(\alpha,\varrho) + \mathtt{T}(\alpha,\varrho)$$

$$\mathtt{h}(\alpha,\varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$$

$$\mathtt{T}(\alpha,\varrho) \equiv i\frac{\varrho}{\sqrt{2}}\left[\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)\right] - \frac{3}{4}\varrho^2$$

$$[\mathtt{H}(\alpha,\varrho), \mathtt{h}(\alpha,\varrho)] = 0$$

$$[\mathtt{H}(\alpha,\varrho), \mathtt{T}(\alpha,\varrho)] = 0$$

$$[\mathtt{H}(\alpha,\varrho), \mathtt{L}] \neq 0$$

TO FIND a COMPLETE O.N. SET of

EIGENSTATES of $\mathtt{H}(\alpha,\varrho)$

SEARCH for COMMON EIGENSTATES of

$2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$ and $\mathtt{T}(\alpha,\varrho)$

- SAE of $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$ have DISCRETE SPECTRUM

- $\mathtt{T}(\alpha, \varrho)$ has CONTINUOUS SPECTRUM $\Rightarrow$

- ANY SAE of $\mathtt{H}(\alpha, \varrho)$ has CONTINUOUS SPECTRUM

$$\Updownarrow$$

ALL THE EIGENSTATES MUST BE IMPROPER

$$\Updownarrow$$

PROPER BOUND STATES ARE FORBIDDEN

$$\Updownarrow$$

THERE IS A **UNIQUE**

ESSENTIALLY SELF-ADJOINT HAMILTONIAN

$$\hat{\mathtt{H}}(\alpha, \varrho) = \hat{\Delta}(\alpha, \varrho) + \mathtt{T}(\alpha, \varrho)$$

• THE SOLUTION •

$$\hat{H}(\alpha,\varrho)\,|n,p_\perp;\hat{\alpha},\varrho\rangle = \left(2n - 2\alpha - \varrho p_\perp - \frac{1}{4}\varrho^2\right)|n,p_\perp;\hat{\alpha},\varrho\rangle$$

$$|n,p_\perp;\hat{\alpha},\varrho\rangle \equiv \sum_{k=0}^{\infty} c_k^{(n)}(\tilde{p})\,|n,k;\hat{\alpha},\varrho\rangle$$

$$c_k^{(n)}(\tilde{p}) = (-i)^k u_k(\tilde{p}) \qquad \tilde{p} \equiv p_\perp - \frac{1}{2}\varrho \qquad \forall n+1 \in \mathbf{N}$$

$$|n,k;\hat{\alpha},\varrho\rangle = \sqrt{\frac{\Gamma(1-\alpha)}{(k!)\Gamma(n+1-\alpha)}}[\bar{\delta}_\varrho(\alpha)]^n[\bar{\theta}_\varrho(\alpha)]^k\,|0,0;\hat{\alpha},\varrho\rangle$$

$$\langle z\bar{z}|0,0;\hat{\alpha},\varrho\rangle = \frac{\gamma^{|\alpha|/2}\exp\{-\frac{1}{4}(|z-\varrho|^2-\varrho^2)\}}{\sqrt{2\pi\Gamma(1-\alpha)\ {}_1F_1(1-\alpha,1;\varrho^2/2)}}$$

$$\langle n,p_\perp;\hat{\alpha},\varrho|J_H|n,p_\perp;\hat{\alpha},\varrho\rangle = -\frac{e^2}{h\Gamma_L}E_H$$

4. CONCLUSION

- ○ MODEL OF DISORDER : POINT-LIKE VORTICES
- ○ 1-IMPURITY HALL PROBLEM EXACTLY SOLVABLE
- ○ WITHOUT HALL FIELD $\mathbf{E}_H = 0 \quad \Rightarrow \quad \infty$ # of SAE
- ○ WHEN $\mathbf{E}_H \neq 0$ THERE IS A **UNIQUE SAE**
- ○ THE P-J PICTURE IS **NOT REALIZED**
- ○ THE HALL CONDUCTANCE IS THE **VERY SAME**
- ○ Σ_{xy} TOPOLOGICAL INVARIANT / C.F.T.