NON-PERTURBATIVE AHARONOV-BOHM EFFECT AND NON-RELATIVISTIC CHERN-SIMONS GAUGE FIELD THEORY

Paola Giacconi & R.S.

Dipartimento di Fisica "A. Righi", Università di Bologna

Fabio Maltoni

Dipartimento di Fisica, Università di Torino

0. INTRODUCTION

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3. CONTACT INTERACTING ANYONS

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Vietri sul mare - March '99

0. INTRODUCTION

• The classical Hamiltonian of two planar particles of mass m interacting *via* "statistical" Aharonov-Bohm (AB) force is

$$H_{\mathrm{cl}} = \frac{1}{2m} \sum_{i=1,2} \left\{ \mathbf{p}_i - \mathbf{A}(\mathbf{r}_i) \right\}^2$$

where the AB potential is $\qquad (\varepsilon_{12} = 1,\ j, k = 1, 2)$

$$A_j\left(\mathbf{r}_1\right) = \frac{e\phi}{2\pi c} \varepsilon_{jk} \frac{\left(\mathbf{r}_1 - \mathbf{r}_2\right)_k}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^2}$$

in which ϕ is the magnetic FLUX

• AB FIELD STRENGTH : infinitely thin solenoyd

$$B = F_{12} = \varepsilon_{jk} \partial_j A_k = \phi \delta^{(2)}\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

• After separation[1] of center of mass and relative motions the classical Hamiltonian becomes

$$\begin{aligned} H_{\mathrm{cl}} &= \frac{\mathbf{P}^2}{4m} + \frac{1}{m} \left\{ \mathbf{p} - \mathbf{A}(\mathbf{r}) \right\}^2 \\ &= H_{\mathrm{c.m.}} + H_{\mathrm{rel}} \end{aligned}$$

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[1] $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,\ 2\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2,\ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

• Quantum mechanical (QM) problem : set

$$\phi_0 \equiv \frac{hc}{e} \simeq 10^{-4}\ \mathrm{G} \times \mathrm{cm}^2, \quad \alpha \equiv \frac{\phi}{\phi_0}$$

• The differential operator corresponding to the HERMITEAN relative Hamiltonian can be written as $(r^2 \equiv x^2 + y^2)$

$$\hat{H}_{\mathrm{rel}} = \frac{\hbar^2}{m} \left\{ \left(i\frac{\partial}{\partial x} + \alpha \frac{y}{r^2} \right)^2 + \left(i\frac{\partial}{\partial y} - \alpha \frac{x}{r^2} \right)^2 \right\}$$

• When $\alpha \in \mathbf{Z}$ the above differential operator is equivalent to the one with $\alpha = 0$ up to gauge transformations on

SINGLE VALUED WAVEFUNCTIONS

Without loss of generality we set $-1 < \alpha \leq 0$

• To select out a QUANTUM RELATIVE HAMILTONIAN

FIND A SELF-ADJOINT OPERATOR

$\Updownarrow$

SUPPLY THE DIFFERENTIAL OPERATOR $\hat{H}_{\mathrm{rel}}$ WITH

BOUNDARY CONDITIONS ON THE WAVEFUNCTION

• The above procedure leads to the genuine QM concept of

CONTACT INTERACTION

1. CONTACT INTERACTION

- $\alpha = 0$: two CLASSICAL FREE PARTICLES

◦ QM LEADS TO THE CONTACT INTERACTION

`Enrico Fermi (1936)`

- *1+1-D* : contact interaction ⇔ **δ-like potential**
- *2+1-D* and *3+1-D* ⇔ **no longer true**

◦ THE CORRECT MATHEMATICAL FRAMEWORK IS

FIND ALL THE SELF-ADJOINT EXTENSIONS

(**SAEs**) of the Hamiltonian hermitean differential operator

`S.Albeverio,F.Gesztesy,R.Hoegh-Krohn,H.Holden(1988)`

- When $\alpha = 0$ the relative Hamiltonian becomes

$$\hat{H}_{\mathrm{rel}}(\alpha = 0) = \frac{\mathbf{p}^2}{m} = -\frac{\hbar^2}{m}\triangle_2$$

where $\triangle_2$ is the *2-D* Laplacian

- To find ALL the SAEs of $\hat{H}_{\rm rel}$ use

◦ BASIC THEOREM

$\hat{H}_{\rm rel}$ **is self-adjoint iff it admits a complete ortho-normal set**

◦ VON NEUMANN'S METHOD

find deficiency indices and subspaces of $\hat{H}_{\rm rel}$

- Follow here below the first option

- Let us look for the MOST GENERAL SOLUTIONS of

$$\hat{H}_{\rm rel}(\alpha = 0)\Psi_E(\mathbf{r}) = E\Psi_E(\mathbf{r})$$

◦ CONTINUOUS SPECTRUM (scattering states)

◦ DISCRETE SPECTRUM (bound states)

• **SCATTERING STATES** : $E = (\hbar^2 k^2/m), \ k > 0$

∘ Since $\left[\hat{H}_{\rm rel}(\alpha = 0), \hat{L}\right] = 0$ we can set

$$\Psi_E(r, \theta) = \sum_{l=-\infty}^{+\infty} \frac{\exp\{il\theta\}}{\sqrt{2\pi}} \psi_l(kr)$$

• The MOST GENERAL solutions of the radial eqs. are

$$\psi_l(kr) = A_l(k) J_l(kr) + B_l(k) N_l(kr) \qquad (l \in \mathbf{Z})$$

where $A_l(k)$ and $B_l(k)$ can be chosen to be real

• Ortho-normality of the IMPROPER eigenfunctions

$$\mathcal{S}' - \lim_{R\to\infty} \int_0^R r dr \psi_l(kr) \psi_{l'}(k'r) = \delta_{l,l'} \delta(k - k') \qquad (\spadesuit)$$

REQUIRES $\psi_l \in H^2_{\rm loc}(\mathbf{R}^2)$

$\Updownarrow$

LOCALLY SQUARE INTEGRABLE (LSI)

- Condition (♠)

$$\mathcal{S}' - \lim_{R\to\infty}\int_0^R r dr \psi_l(kr)\psi_{l'}(k'r) = \delta_{l,l'}\delta(k-k') \qquad (\spadesuit)$$

is fulfilled **if and only if**

$$\psi_l(kr) = \sqrt{k} J_l(kr) \qquad (\forall l \neq 0)$$
$$B_l(k) \equiv 0 \qquad (\forall l \neq 0)$$

$$\psi_0(kr; E_0) = A_0(k) J_0(kr) + B_0(k) N_0(kr) \qquad (l = 0)$$

$$\frac{B_0(k)}{A_0(k)} = -\frac{\pi}{\ln(\hbar^2 k^2/m|E_0|)} \qquad (l = 0)$$

E_0 is some suitable ENERGY SCALE

- The S-wave $\psi_0(kr; E_0)$ is SINGULAR AT $r = 0$

$$N_0(kr) \sim \frac{\pi}{2} \ln\frac{kr}{2} \qquad (r \sim 0)$$

• **BOUND STATE** : $E = E_B = -(\hbar^2\kappa^2/m) < 0$

The radial Schrödinger equation for the S-wave allows the NEGATIVE ENERGY eigenfunction (normalized to one)

$$\psi_B(\kappa r) = \sqrt{(\kappa/\pi)}K_0(\kappa r)$$

• The bound state is PRESENT iff

$$\int_0^\infty r dr \psi_0(kr; E_0)\psi_B(\kappa r) = 0$$

which is TRUE PROVIDED

$$E_B = -(\hbar^2\kappa^2/m) \equiv E_0 < 0$$

• To sum up ORTHO-NORMALITY and COMPLETENESS

$$\Updownarrow$$

THEOREM

ALL THE SAEs of $\hat{H}_{\rm rel}(\alpha = 0)$
ARE A ONE-PARAMETER FAMILY
labelled by the energy scale
$-\infty \leq E_0 < 0$

○ ONLY the case $E_0 = -\infty$ corresponds to REGULAR EIGENFUNCTIONS i.e. $B_l \equiv 0$ ($\forall l \in \mathbf{Z}$) that is to a FREE PARTICLE (Friedrichs' extension, **scale invariant**)

○ In ALL the other cases $-\infty < E_0 < 0$ the radial S-wave eigenfunctions are ALWAYS singular LSI at the origin and a BOUND STATE ALWAYS EXISTS whose energy is E_0

○ The non-trivial SCATTERING AMPLITUDE is

$$\sqrt{\pi i k} f(k, E_0) = \frac{2\pi i}{\ln(-\hbar^2 k^2 / m E_0) - i\pi}$$

2. NON-RELATIVISTIC FIELD THEORY

$(\hbar = c = 1)$

- QUESTION : is it possible to identify the QM scale

$$\sqrt{m|E_B|} \quad : \quad \text{contact interaction scale}$$

and the SUBTRACTION POINT μ
of some renormalizable field theory?

○ ANSWER : **YES** `Oren Bergman : PRD46 (1992)`

- *2+1-D* 1PI renormalized 4-point function in CMF

$$\sqrt{\pi i p} f(p,\theta) = (m/4)\Gamma_{\mathrm{R}}^{(4)}(p,\theta) \qquad (\clubsuit)$$

- Renormalized lagrangean density in *2ω+1-D*

$$\mathcal{L}_R \equiv \mathcal{L} \ + \ \mathcal{L}_{\text{c.t.}} = \phi^* \left(i\partial_t + \frac{1}{2m}\triangle_2 \right) \phi - \mu^{2\epsilon} \frac{\lambda_0}{4m} (\phi^* \phi)^2$$

where $\epsilon \equiv \omega - 1$ and μ is the dimensional regularization mass

- The ONLY DIVERGENT QUANTITY IS

$$\lambda_0 \equiv a_0(\lambda_{\text{MS}}) + \sum_{k=1}^{\infty} a_k(\lambda_{\text{MS}}) \frac{1}{\epsilon^k}$$

λ_{MS} is the MS-scheme RENORMALIZED COUPLING

• EXACT RENORMALIZED 1PI 4-point function :

$$\Gamma_{\mathrm{R}}^{(4)}(p/\mu') = \left(-i\frac{\lambda_{\mathrm{MS}}}{m}\right)\frac{1}{1-(\lambda_{\mathrm{MS}}/4\pi)\left\{\ln(p/\mu')-i(\pi/2)\right\}}$$

$$\ln\mu' = \ln\mu + (1/2)[\ln(4\pi)-\gamma_E]$$

$$0 \le \frac{\lambda_{\mathrm{MS}}(\mu')}{4\pi}\ln\frac{p}{\mu'} < 1$$

• According to (♣) we get

$$\frac{4\pi}{\lambda_{\mathrm{MS}}} = \ln\left(\frac{\sqrt{-mE_B}}{\mu'}\right) \qquad (\diamond)$$

$$\mu'_{\mathrm{phys}} = \sqrt{-mE_B} \quad \Leftrightarrow \quad \text{pole in } \lambda_{\mathrm{MS}}$$

$$\Updownarrow$$

$\alpha = 0$ CORRESPONDENCE between QM and NRQFT

$(-\infty \le E_B < 0)$

- **REMARKS** : $(\diamond)\ \frac{4\pi}{\lambda} = \ln\left(\frac{\sqrt{-mE_B}}{\mu'}\right)$

◦ EXACT β-function :

$$\beta(\lambda) = \frac{\lambda^2}{4\pi}$$

◦ SCALE INVARIANCE :

$$\beta = 0 \Leftrightarrow \lambda = 0 \Leftrightarrow E_B = -\infty$$

◦ NATURE OF CONTACT INTERACTION :

$$\text{contact interaction} \neq \lambda(\hbar^2/m)\delta^{(2)}(\mathbf{r})$$

NO LINK BETWEEN THE SIGN OF λ
AND ATTRACTIVE/REPULSIVE NATURE
OF CONTACT INTERACTION IN *2D*

$$\Updownarrow$$

2-D CONTACT INTERACTION **ALWAYS ATTRACTIVE**
(THE BOUND STATE IS ALWAYS PRESENT)

3. CONTACT INTERACTING ANYONS

- Go back to the AB relative Hamiltonian

$$\hat{H}_{\rm rel}(\alpha) = \frac{1}{m}\left\{\mathbf{p} - \alpha\frac{\hat{\mathbf{k}}\wedge\mathbf{r}}{r^2}\right\}^2$$

describing ANYONS of STATISTICAL WEIGHT $-1 < \alpha < 0$

- "BOSE-MADE" anyons $\Leftrightarrow$ the wavefunction fulfils

$$\Psi_{\rm rel}(\mathbf{r}) = \Psi_{\rm rel}(-\mathbf{r})$$

i.e. writing again

$$\Psi_E(r,\theta) = \sum_{j=-\infty}^{+\infty}\frac{\exp\{2ij\theta\}}{\sqrt{2\pi}}\psi_{2j}(kr)$$

we restrict to EVEN VALUED angular momenta

- Search for the SAEs of the relative Hamiltonian (♡)

∘ $O(2)$-INVARIANT $\Leftrightarrow [\hat{H}_{\rm rel}(\alpha; E_0), L] = 0$

∘ BOSE SYMMETRIC $\Leftrightarrow \Psi_{\rm rel}(\mathbf{r}) = \Psi_{\rm rel}(-\mathbf{r})$

• **SCATTERING STATES** : $E = (\hbar^2 k^2/m),\ k > 0$

$$\psi_l(kr) = A_l(k) J_{l-\alpha}(kr) + B_l(k) N_{l-\alpha}(kr) \quad (l \text{ even})$$

• Ortho-normality of the IMPROPER eigenfunctions (♠)

$$\psi_l(kr) = \sqrt{k} J_{l-\alpha}(kr) \qquad (\forall l \neq 0)$$
$$B_l(k) \equiv 0 \qquad (\forall l \neq 0)$$

$$\psi_0(kr; E_0) = A_0(k,\alpha) J_{|\alpha|}(kr) + B_0(k,\alpha) N_{|\alpha|}(kr) \qquad (l = 0)$$

$$\frac{B_0(k,\alpha)}{A_0(k,\alpha)} = \frac{\sin \pi\alpha}{\cos \pi\alpha + \mathrm{sgn}(E_0) \left(\hbar^2 k^2/m|E_0|\right)^{\alpha}} \qquad (l = 0)$$

$-\infty \leq E_0 < +\infty$ is some suitable ENERGY SCALE

• The S-wave $\psi_0(kr; E_0)$ is SINGULAR AT $r = 0$

$$N_{|\alpha|}(kr) \sim -\frac{\csc \pi|\alpha|}{\Gamma(1-|\alpha|)} \left(\frac{kr}{2}\right)^{-|\alpha|} \qquad (r \sim 0)$$

• **BOUND STATE** : $E = E_B = -(\hbar^2\kappa^2/m) < 0$

The radial Schrödinger equation for the S-wave allows the NEGATIVE ENERGY eigenfunctions (normalized to one)

$$\psi_B(\kappa r) = \sqrt{(\kappa/\pi)}K_\alpha(\kappa r)$$

• The bound state is PRESENT iff

$$\int_0^\infty r dr \psi_0(kr; E_0)\psi_B(\kappa r) = 0$$

which is TRUE PROVIDED $E_B = -(\hbar^2\kappa^2/m) \equiv E_0 < 0$

BUT NOT TRUE when $E_0 \geq 0$

• Ortho-normality and completeness entail the following

THEOREM

The (♡) SAEs of $\hat{H}_{\rm rel}(\alpha)$
ARE A ONE-PARAMETER FAMILY
labelled by the energy scale

$$-\infty \le E_0 < +\infty$$

∘ ONLY the case $E_0 = -\infty$ corresponds to REGULAR EIGENFUNCTIONS i.e. $B_l \equiv 0$ $(\forall l \in \mathbf{Z})$ that is the case of the original Aharonov-Bohm Hamiltonian

(FREE ANYONS)

`Y. Aharonov, D. Bohm :  PRD115 (1959)`

∘ In ALL the other cases $-\infty < E_0 < +\infty$ the radial S-wave eigenfunctions are ALWAYS SINGULAR LSI at the origin and a BOUND STATE EXISTS ONLY WHEN $-\infty < E_0 < 0$

(CONTACT INTERACTING ANYONS)

∘ The S-wave SCATTERING AMPLITUDE is $\qquad (p \equiv \hbar k)$

$$f_0(p, E_0; \alpha) = \frac{\left(1 - e^{i\pi\alpha}\right)}{\sqrt{\pi i p}} \, \frac{1 - \mathrm{sgn}(E_0)(p^2/m|E_0|)^\alpha}{\exp\{i\pi\alpha\} + \mathrm{sgn}(E_0)(p^2/m|E_0|)^\alpha}$$

• **REMARKS** :

◦ Spectral decomposition of (♡) SAEs $\hat{H}_{\rm rel}(\alpha; E_0)$

$$\hat{H}_{\rm rel}(\alpha; E_0) = \sum_{j=-\infty}^{+\infty} \int_0^\infty dk \, \frac{\hbar^2 k^2}{m} \, |2j, k\rangle \, \langle k, 2j| + \vartheta(-E_0) \, |B\rangle \, \langle B|$$

$$\langle r, \theta | 2j, k\rangle = \frac{\exp\{2ij\theta\}}{\sqrt{2\pi}} \psi_{2j}(kr; \alpha, E_0)$$

$$\langle r|B\rangle = \psi_B(\kappa r) = \sqrt{(\kappa/\pi)} K_\alpha(\kappa r)$$

◦ In the limit $\alpha \to 0$ we recover the one-parameter family of the pure contact interaction studied in the previous sections :

$$\lim_{\alpha \to 0} f_0(p, E_0; \alpha) = f(k, E_0) = \frac{2\pi}{\sqrt{\pi i k}} \frac{i}{\ln(-\hbar^2 k^2/mE_0) - i\pi}$$

◦ The most general family of SAEs of the AB Hamiltonian is a FOUR-PARAMETER CONTINUOUS FAMILY

`R. Adami, S. Teta (1997)`

4. CHERN-SIMONS GAUGE FIELD THEORY

- QUESTION : can the EXACT QM scattering amplitude

$$f_0(p,E_0;\alpha)=\frac{\left(1-e^{i\pi\alpha}\right)}{\sqrt{\pi i p}}\,\frac{1-\mathrm{sgn}(E_0)(p^2/m|E_0|)^\alpha}{\exp\{i\pi\alpha\}+\mathrm{sgn}(E_0)(p^2/m|E_0|)^\alpha}$$

BE RECOVERED from PERTURBATIVE NRQFT ?

- ANSWER : NO it is not possible

`P. Giacconi, F. Maltoni, R.S. :  PLB (1998)`

CONTACT INTERACTING ANYONS ARE

TRULY NON-PERTURBATIVE OBJECTS

STUDY PERTURBATIVE RENORMALIZABLE NR CHERN-SIMONS GAUGE FIELD THEORY COUPLED TO SCALAR CHARGED MATTER FIELDS IN *2+1-D*

• Renormalized lagrangean density in *2ω+1-D*

$$\mathcal{L}_R = \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{CS}}$$

$$\mathcal{L}_{\mathrm{matter}} = \phi^* \left(i\partial_t + eA_0 \right) \phi + \frac{1}{2m} \left| i\nabla\phi + e\mathbf{A}\phi \right|^2 - \mu^{2\epsilon} \frac{\lambda_0}{4m} (\phi^* \phi)^2$$

$$\mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{2} \varepsilon_{jl} A_l \left(\partial_t A_j - \partial_j A_0 \right)$$

• Divergent bare scalar self-coupling ($\lambda \equiv \lambda_{\mathrm{MS}}$)

$$\lambda_0 \equiv a_0(\lambda, \nu) + \sum_{k=1}^{\infty} a_k(\lambda, \nu) \frac{1}{\epsilon^k}$$

• Finite CS minimal coupling and AB flux intensity

$$\nu \equiv \frac{e^2}{\kappa} = 2\pi\alpha$$

• From Feynman's rules we get

$$
\begin{aligned}
&\Gamma^{(4)}_{\mathrm{R},s}(p/\mu',\nu)\Big|_{2-loop} = \\
&- i\frac{\lambda}{m} - \frac{\nu^2}{2m} - \frac{i}{m}\frac{\lambda^2-4\nu^2}{4\pi}\left(\ln\frac{p}{\mu'} - i\frac{\pi}{2}\right) \\
&- \frac{i\lambda}{4\pi m}\frac{\lambda^2-4\nu^2}{4\pi}\left(\ln\frac{p}{\mu'} - i\frac{\pi}{2}\right)^2 + i\frac{\lambda\nu^2}{24m}
\end{aligned}
$$

$\Gamma^{(4)}_{\mathrm{R},s}(p/\mu',\nu)$ is the S-wave component of the renormalized 1PI 4-point function in the MS-scheme

• The bare and renormalized scalar self-coupling are

$$\lambda_0(\epsilon) = \lambda + \frac{\lambda^2-4\nu^2}{8\pi}\frac{1}{\epsilon} + \frac{\lambda}{8\pi}\frac{\lambda^2-4\nu^2}{8\pi}\frac{1}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

• COMPARE NOW : $[(\clubsuit)\ \sqrt{\pi i p}\, f(p,\theta) = (m/4)\Gamma^{(4)}_{\mathrm{R}}(p,\theta)]$

$\Gamma^{(4)}_{\mathrm{R},s}(p/\mu',\nu)\Big|_{2-loop}$ and α-POWER SERIES of $f_0(p,E_0;\alpha)$

• **RESULT I** :

◦ WHEN $0 < E_0 < +\infty$ $\Leftrightarrow$ ABSENCE OF BOUND STATES

$$\Gamma^{(4)}_{\mathrm{R},s}(p/\mu', \nu)\Big|_{2-loop} \text{ and } \alpha\text{-POWER SERIES of } f_0(p, E_0; \alpha)$$

MATCH PROVIDED WE IDENTIFY

$$\mu' = \sqrt{mE_0}$$
$$\lambda_{\mathrm{MS}} = 0$$

PURE MINIMALLY COUPLED
CS GAUGE FIELD THEORY

• **RESULT II** :

○ WHEN $-\infty < E_0 < 0$ $\Leftrightarrow$ PRESENCE OF BOUND STATES

$$\sqrt{\pi i p} f_0(p, E_B; \alpha) \simeq$$

$$\frac{(im/4)T(p,E_B)}{1-(im/8)T(p,E_B)} + \frac{4i\pi^2\alpha^2}{3mT(p,E_B)} \frac{1+(im/8)T(p,E_B)}{1-(im/8)T(p,E_B)}$$

$$T(p,E_B) \equiv \frac{(8\pi/m)}{\ln(-p^2/mE_B)}$$

○ RED TERM IS NOTHING BUT

$$\sqrt{\pi i p} f_0(p, E_B; \alpha = 0) = \frac{2\pi i}{\ln(-p^2/mE_0) - i\pi}$$

and consequently from Bergman's result

$$T(p,E_B) = \frac{(\lambda/m)}{1-(\lambda/4\pi)\ln(p/\mu')}$$

$$\Updownarrow$$

THE COEFFICIENT OF α^2 IS

NOT ANALYTIC IN λ

EXACT QM & PQFT CAN NOT MATCH

5. CONCLUSION

• CONTACT INTERACTING BOSE-MADE ANYONS
PROVIDE A VERY NICE EXAMPLE OF
GENUINE NON-PERTURBATIVE EFFECTS

• GENERALIZATION TO FERMI-MADE ANYONS
FAR FROM BEING TRIVIAL (IN PROGRESS)

• NON-RELATIVISTIC LIMIT SUGGESTS
NON-TRIVIALITY OF $\lambda\,(\phi^{*}\phi)_{4}^{2}$
(IN SPITE OF LATTICE SIMULATIONS)