

**BOSE-EINSTEIN CONDENSATION
IN THE PRESENCE OF ONE IMPURITY
AND A UNIFORM FIELD**

Ricardo Moritz Cavalcanti

Instituto de Fisica, Universidade Federal do Rio de Janeiro

Fabio Maltoni

Department of Physics, Urbana-Champaign

Guido Pupillo

Laboratory for Physical Science, College Park

Paola Giacconi & R. S.

Dipartimento di Fisica, Università di Bologna

0. INTRODUCTION

1. POINT-LIKE IMPURITY

2. BOSE-EINSTEIN CONDENSATION

3. UNIFORM FIELD

4. CONCLUSIONS

Trento - Dicembre 2001

0. INTRODUCTION

- **Ideal Quantum Systems in D Spatial Dimensions in the presence of External Fields**
 - Gravity-like Uniform Fields
 - crossed Hall's Fields
 - Point-like Disorder : *AB-vortex & δ -like Interaction*
- *δ -like Potentials* are well defined only in $D = 1$
 $D = 2, 3$ **Renormalization** \Leftrightarrow **Contact Interaction**
- **Solvable Mathematical Models**
Self-Adjoint Exts of Symmetric Hamiltonians
 - *von Neumann's Method of Deficiency Indices*
 - *Krein's Formula for the Resolvent*



Contact Interaction

“ ... If the particles of a 3D ideal boson gas were placed in a uniform gravitational field, then Bose-Einstein condensation would still occur, but in the condensation region there would be a spatial separation of the two phases, just as in a gas-liquid condensation ... ”

pg. 266

K. Huang, *Statistical Mechanics* (Wiley, New York, 1987)

long standing popular belief

L. Goldstein, *J. Chem. Phys.* **9**, 273 (1941)

W. Lamb and A. Nordsieck, *Phys. Rev.* **59**, 677 (1941)

S. R. de Groot, G. J. Hooyman and C. A. ten Seldam,
Proc. R. Soc. London A **203**, 266 (1950)

O. Halpern, *Phys. Rev.* **86**, 126 (1952); **87**, 520 (1952).

H. A. Gersch, *J. Chem. Phys.* **27**, 928 (1957)

V. Bagnato, D. E. Pritchard and D. Kleppner, *Phys. Rev.*
A **35**, 4354 (1987)

etc.

Theorem

Suppose the one-particle energy spectrum of an ideal boson gas satisfies the following conditions

- (i) there is a gap between the fundamental and the first excited energy levels

$$E_1 - E_0 = \Delta > 0$$

- (ii) the one-particle partition function is finite

$$Z \equiv \sum_{n=0}^{\infty} d_n \exp(-\beta E_n) < \infty$$

d_n being the finite degeneracy of the n -th eigenvalue of the one-particle Hamiltonian

Then this gas displays Bose-Einstein condensation

Proof

If $\mu < E_0$ the No. of particles in the excited states is bounded

$$N_{\text{ex}} = \sum_{n=1}^{\infty} \frac{d_n \exp[-\beta(E_n - \mu)]}{1 - \exp[-\beta(E_n - \mu)]}$$
$$\leq \frac{\exp(\beta\mu)}{1 - \exp[-\beta(E_1 - \mu)]} \sum_{n=1}^{\infty} d_n \exp(-\beta E_n)$$

Therefore

$$\lim_{\mu \rightarrow E_0} N_{\text{ex}} \leq \frac{\exp(\beta E_0)}{1 - \exp(-\beta\Delta)} [Z - d_0 \exp(-\beta E_0)] < \infty$$

since by hypothesis $\Delta > 0$ and Z and d_0 are finite *QED*

generalization to

continuous spectrum & degeneracy

under the suitable introduction of

1-particle partition function *per* unit volume

density of particles in the excited states

1. POINT-LIKE IMPURITY

- Classical point-like impurity in 2D : AB-vortex
(Aharonov & Bohm 1959)

$$H(\alpha) = \frac{1}{2m} \left[\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right]^2$$

$$A_j(x_1, x_2) = \frac{\phi}{2\pi} \epsilon_{jk} \frac{x_k}{r^2}$$

$$\alpha = \frac{\phi}{\phi_0} = \frac{e\phi}{hc} \quad \phi_0 \equiv \frac{hc}{e} \sim 10^{-4} \text{Gauss} \times \text{cm}^2$$

$$\mathbf{p}, \mathbf{r} \in \mathbf{R}^2 \quad r \equiv \sqrt{x_1^2 + x_2^2}$$

- single valuedness $\implies -1 < \alpha \leq 0$

* quantization of classical Hamiltonian *

* leads to **SYMMETRIC** operator *



FIND ALL THE SELF-ADJOINT EXTENSIONS



CONTACT INTERACTION (CI)

in QUANTUM MECHANICS

(Fermi 1936)

⊙ $D = 1$ **CI fully equivalent** to δ -potential

⊙ $D > 1$ **CI NOT equivalent** to δ -potential

(Albeverio 1988)

- **Search $O(2)$ -invariant Self-Adjoint Extensions**

solve the radial problem of angular momentum l

$$\odot \quad \psi_l(kr) = A_l J_{|l+\alpha|} + B_l N_{|l+\alpha|}$$

To fix A_l and B_l

\Updownarrow

impose boundary conditions

at the impurity position $r = 0$

\Updownarrow

Self-Adjointness of the Hamiltonian operator

- square integrability around impurity position

$$B_l = 0 \quad \Leftrightarrow \quad l \neq 0$$

- \odot Let us fix the **S -wave** coefficients A_0 and B_0

Continuous Spectrum

- Eigenvalues $E = (\hbar^2 k^2 / 2m)$ $k > 0$

- Improper Eigenfunctions

$$\psi_l(k; r, \theta) = \sqrt{\frac{k}{2\pi}} e^{il\theta} J_{|l+\alpha|}(kr) \quad l \in \mathbf{Z} - \{0\}$$

- $\psi_0(k, E_0; r) = A_0(k; \alpha, E_0) J_{|\alpha|}(kr) + B_0(k; \alpha, E_0) N_{|\alpha|}(kr)$

★ $E_0 \in \mathbf{R}$ is some suitable **physical energy scale** ★

- Small distance behaviour

$$N_{|\alpha|}(kr) \sim \frac{\csc(\pi\alpha)}{\Gamma(1 - |\alpha|)} \left(\frac{kr}{2}\right)^{-|\alpha|} \quad (r \sim 0)$$

- **Orthonormality**

$$\lim_{R \rightarrow \infty} \int_0^R 2\pi dr r \psi_0(k, E_0; r) \psi_0^*(k', E_0; r) = \delta(k - k')$$

\Updownarrow

$$\frac{B_0(k; \alpha, E_0)}{A_0(k; \alpha, E_0)} = \frac{\sin(\pi\alpha)}{\cos(\pi\alpha) + \text{sgn}(E_0) (\hbar^2 k^2 / 2m |E_0|)^\alpha}$$

- **Regular solution** \Leftrightarrow Aharonov-Bohm wave function

$$B_0(k; \alpha, E_0) = 0 \Leftrightarrow E_0 \rightarrow \pm\infty$$

\Updownarrow

★ **no Contact Interaction** ★

- **Special cases**

$|\alpha| = 0$ **pure Contact Interaction in 2D**

$|\alpha| = \frac{1}{2}$ **pure Contact Interaction in 3D**

Discrete Spectrum

Completeness \Leftrightarrow one bound state exists iff $E_0 < 0$

- $\langle r | \psi_B \rangle = \psi_B(\kappa, r) = \frac{\kappa}{\pi} \sqrt{\frac{\sin(\pi\alpha)}{\alpha}} K_\alpha(\kappa r)$

$$\hbar\kappa \equiv \sqrt{2m|E_0|}$$

\Updownarrow

Self-Adjoint $O(2)$ -invariant Hamiltonians
defined by their **spectral decompositions**

- $$H(\alpha, E_0) = \sum_{l=-\infty}^{+\infty} \int_0^\infty dk \frac{\hbar^2 k^2}{2m} |l, k\rangle \langle k, l|$$
$$+ \vartheta(-E_0) E_0 |\psi_B\rangle \langle \psi_B|$$

- $$\langle r, \theta | l, k \rangle = \frac{\exp\{il\theta\}}{\sqrt{2\pi}} \psi_l(k, r; \alpha, E_0) \quad k \geq 0$$

★ REMARKS ★

- **Physical meaning** of the quantum energy scale E_0

$$E_0 < 0 \quad \Longrightarrow \quad \text{bound state energy}$$

$$E_0 > 0 \quad \Longrightarrow \quad \text{resonance energy}$$

$$\delta_l(\alpha) = \frac{\pi}{2} |\alpha| \operatorname{sgn}(l) = \frac{\pi}{2} [|l| - |l + \alpha|] \quad l \neq 0$$

$$\delta_0(k; \alpha, E_0) = \frac{\pi}{2} \alpha - \operatorname{arctg} \left[\frac{\sin(\pi\alpha)}{\cos(\pi\alpha) + \operatorname{sgn}(E_0)(k^2/|E_0|)^\alpha} \right]$$

- $H(\alpha, E_0)$ is a **continuous one-parameter family** of Self-Adjoint Extensions of the symmetric Hamiltonian $H(\alpha)$ labelled by the physical quantum energy scale E_0 (von Neumann's theorem)

★ **FUNDAMENTAL THEOREM** ★

Self-Adjointness



Orthonormality & Completeness



von Neumann's method of deficiency indices

Special cases

- $|\alpha| = 0$ **pure Contact Interaction in 2D**

$$H_{2D}(E_0) = \sum_{l=-\infty}^{+\infty} \int_0^{\infty} dk \frac{\hbar^2 k^2}{2m} |l, k\rangle \langle k, l| \\ + E_0 |\psi_B\rangle \langle \psi_B| \quad [E_0 < 0]$$

★ **the bound state always exists** ★

- $|\alpha| = \frac{1}{2}$ **pure Contact Interaction in 3D**

$$H_{3D}(E_0) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \int_0^{\infty} dk \frac{\hbar^2 k^2}{2m} |l, m, k\rangle \langle k, m, l| \\ + \vartheta(-E_0) E_0 |\psi_B\rangle \langle \psi_B| \quad [E_0 \in \mathbf{R}]$$

★ **dimensional transmutation** ★

★ **Point-like DISORDER** ★

- If the **Domain** of the Hamiltonian operator is that of wave functions **REGULAR** on the hole plane



$$E_0 = \pm\infty \quad H(\alpha, E_0 = \pm\infty) \equiv H_{\text{AB}}(\alpha)$$

absence of Contact Interaction

- If the **Domain** of the Hamiltonian operator is that of wave functions **SINGULAR** at the impurity position



$$E_0 \neq \pm\infty \quad H = H(\alpha, E_0)$$

presence of Contact Interaction

- ⊙ point-like disorder in 2D is described in **QM** by

TWO parameters α, E_0

2. BOSE-EINSTEIN CONDENSATION

- **To discuss BEC of ideal systems we need**
 - the 1-particle partition function

$$Z_{2D}(\alpha, \beta, E_0)$$

- the density of 1-particle states

$$\rho_{2D}(\alpha, E, E_0)$$

- **the average particles density**

$$\langle n(\alpha, \beta, \mu, E_0) \rangle_{2D}$$

$$\beta \equiv \frac{1}{k_B T} \quad z \equiv e^{\beta \mu} \quad \text{fugacity}$$

The 1-particle partition function
in a large 2D box of area A

$$Z_{2D}(\alpha, \beta, E_0) = \frac{A}{\lambda_T^2} + \frac{\alpha(\alpha + 1)}{2} + \vartheta(-E_0)e^{\beta|E_0|}$$
$$+ \frac{\alpha \sin(\pi\alpha)}{\pi} \int_0^\infty \frac{dx}{x^{1+\alpha}} \frac{\operatorname{sgn}(E_0) e^{-\beta|E_0|x}}{1 + 2\operatorname{sgn}(E_0)x^{|\alpha|} \cos(\pi\alpha) + x^{2|\alpha|}}$$

$$\lambda_T \equiv \frac{h}{\sqrt{2\pi m k_B T}}$$

Infrared Divergence $A \rightarrow \infty$

only in the free part

Special Cases

- $|\alpha| = 0$ **pure CI in 2D** $[E_0 < 0]$

$$\begin{aligned} Z_{2D}(\beta, E_0) &= \frac{A}{\lambda_T^2} + e^{\beta|E_0|} - \int_0^\infty \frac{dE}{E} \frac{e^{-\beta E}}{\ln^2(-E/E_0) + \pi^2} \\ &= \frac{A}{\lambda_T^2} + \nu(\beta|E_0|) \end{aligned}$$

$$\nu(x) \equiv \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt$$

the bound state is always present

- $|\alpha| = \frac{1}{2}$ **pure CI in 3D** $[E_0 \in \mathbf{R}]$

$$\begin{aligned} Z_{3D}(\beta, E_0) &= \frac{V}{\lambda_T^3} + \vartheta(-E_0)e^{\beta|E_0|} \\ &\quad + \frac{1}{2} \operatorname{sgn}(E_0)e^{\beta|E_0|} \operatorname{erfc}(\sqrt{\beta|E_0|}) \end{aligned}$$

the bound state is present iff $E_0 < 0$

★ **dimensional transmutation** ★

The density of 1-particle states

Laplace inverse transform of Z_{2D}

$$\rho_{2D}(\alpha, E, E_0) = \vartheta(E) \frac{2\pi m}{h^2} A + \frac{\alpha(\alpha + 1)}{2} \delta(E) + \varrho(\alpha, E, E_0)$$

$$\begin{aligned} \varrho(\alpha, E, E_0) &\equiv \frac{\alpha \sin(\pi\alpha)}{\pi E} \vartheta(E) \operatorname{sgn}(E_0) \\ &\times \frac{(E|E_0|)^{|\alpha|}}{E^{2|\alpha|} + |E_0|^{2|\alpha|} + 2\operatorname{sgn}(E_0)(E|E_0|)^{|\alpha|} \cos(\pi\alpha)} \end{aligned}$$

The mean density at thermal equilibrium

$$\begin{aligned} \langle n(\alpha, \beta, \mu, E_0) \rangle_{2D} &= \lambda_T^{-2} g_1(z) + \frac{z\alpha(\alpha + 1)}{2A(1 - z)} \\ &+ \vartheta(-E_0) \frac{z}{A(z_0 - z)} + \vartheta(E_0) \frac{z}{A(1 - z)} \\ &+ \frac{z}{A} \int_0^\infty dE \varrho(\alpha, E, E_0) \frac{e^{-\beta E}}{1 - z \exp\{-\beta E\}} \end{aligned}$$

$$z_0 \equiv \exp\{\beta E_0\} \quad g_1(z) = -\ln(1 - z)$$

★ **Results** ★

- **Thermodynamic limit**

$$A, \langle N \rangle \rightarrow \infty \quad \langle n \rangle_{2D} \equiv \frac{\langle N \rangle}{A} \quad \text{fixed}$$

- **The range of fugacity** $D = 2 \quad \alpha \neq 0$

⊙ $0 \leq z \leq z_0 < 1$ iff $E_0 < 0$

⊙ $0 \leq z < 1$ iff $E_0 \geq 0$

**BEC occurs only in the presence of the bound state
only for the sub-family of the Self-Adjoint
Extensions of the symmetric Hamiltonian**

$$-\infty < E_0 < 0$$

- **Critical quantities does not depend upon α**

$$\ln(1 - e^{\beta_c E_0}) = -\frac{h^2 \beta_c}{2\pi m} \langle n \rangle_{2D}$$

Pure Contact Interaction in 2D

$$|\alpha| = 0 \quad E_0 < 0$$

$$\langle n(\beta, \mu, E_0) \rangle_{2D} = \lambda_T^{-2} g_1(z) + \frac{z}{A(z_0 - z)}$$
$$- \frac{z}{A} \int_0^\infty \frac{dE}{E} \frac{e^{-\beta E}}{1 - z \exp\{-\beta E\}} \frac{1}{\ln^2(-E/E_0) + \pi^2}$$



BEC occurs $\forall E_0 < 0$

but for the free case $E_0 = -\infty$

★ Critical quantities ★

$$\ln(1 - e^{\beta_c E_0}) = -\frac{h^2 \beta_c}{2\pi m} \langle n \rangle_{2D}$$

Pure Contact Interaction in 3D

$$|\alpha| = \frac{1}{2} \quad E_0 \in \mathbf{R}$$

$$\begin{aligned} \langle n(\beta, \mu, E_0) \rangle_{3D} &= \lambda_T^{-3} g_{3/2}(z) + \frac{z\vartheta(-E_0)}{V(z_0 - z)} + \frac{z\vartheta(E_0)}{V(1 - z)} \\ &+ \frac{z}{V} \int_0^\infty \frac{dE}{2\pi E} \frac{\operatorname{sgn}(E_0) \sqrt{E} |E_0| e^{-\beta E}}{(E + |(E_0)|)(1 - z \exp\{-\beta E\})} \end{aligned}$$

↕

BEC always exists in 3D

★ **Critical quantities** ★

$$\lambda_{T_c}^3 \langle n \rangle_{3D} = \zeta(3/2) \quad E_0 \geq 0$$

$$\lambda_{T_c}^3 \langle n \rangle_{3D} = g_{3/2}(e^{\beta_c E_0}) \quad E_0 \leq 0$$

$$g_s(z) \equiv \frac{z}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1} \exp\{-\beta x\}}{1 - z \exp\{-\beta x\}}$$

3. UNIFORM FIELD

- **1-particle Hamiltonian in D spatial dimensions**

$$H_0^{(D)}(g) = \frac{\mathbf{p}^2}{2m} + mgx \quad g \geq 0$$

- Notations

$$\mathbf{x} = (x_1, \dots, x_D) \equiv (\mathbf{r}, x) \quad \mathbf{p} = (p_1, \dots, p_D) \equiv (\mathbf{k}, p)$$

- Boson Gas in the half-space $x \geq 0$
- **Hamiltonian bounded from below \Leftrightarrow Stability**
- **Neumann's Boundary Conditions on the Floor**

$$\partial_x \psi(\mathbf{r}, x)|_{x=0} = 0$$

**main conclusions do not depend on
specific boundary conditions on the floor**

Spectrum & Eigenfunctions

- **Regular 1-particle Eigenfunctions**

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{\exp\{(i/\hbar)\mathbf{k}\cdot\mathbf{r}\}}{(2\pi\hbar)^{(D-1)/2}} \sqrt{\frac{-\kappa}{a'_n}} \frac{\text{Ai}(\kappa x + a'_n)}{\text{Ai}(a'_n)}$$

- $\text{Ai}(x)$ is the **Airy's function**

$a'_n < 0$ are the **zeros of $\text{Ai}'(x)$**

- **1-particle Spectrum**

$$E_{n,\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - E_g a'_n \quad n \in \mathbf{N} \quad \mathbf{k} \in \mathbf{R}^{D-1}$$

$$\kappa \equiv \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \quad E_g \equiv \frac{mg}{\kappa} = \frac{\hbar^2 \kappa^2}{2m}$$

- **The spectrum is purely continuous if $D \neq 1$**

★ **Translation Invariance in Transverse Space** ★

average Number of particles

per unit transverse Volume V_{\perp}

- **Thermodynamic limit**

$$V_{\perp}, \langle N \rangle \rightarrow \infty \quad \langle n \rangle_D \equiv \frac{\langle N \rangle}{V_{\perp}} \text{ fixed}$$

- **The range of chemical potential** $-\infty < \mu \leq -E_g a'_1$
- **The mean density of particles in the excited states**

$$\begin{aligned} \langle n_{\text{ex}}(g, \beta, \mu) \rangle_D &= \\ \sum_{j=1}^{\infty} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}} \left\{ \exp \left[\beta \left(\frac{\mathbf{k}^2}{2m} - E_g a'_j - \mu \right) \right] - 1 \right\}^{-1} \\ &= \lambda_T^{1-D} \sum_{j=1}^{\infty} g_{(D-1)/2} \left(\exp \{ \beta E_g a'_j + \beta \mu \} \right) \end{aligned}$$

$$D > 1 \quad \mu < -E_g a'_1$$

Absence of BEC for $D = 2, 3$

Look at the **FIRST** term in the sum

$$\lim_{\mu \uparrow -E_g a'_1} g_{(D-1)/2} (\exp \{ \beta E_g a'_1 + \beta \mu \}) = \infty$$

$$\text{iff} \quad D = 2, 3$$



$$\lim_{\mu \uparrow -E_g a'_1} \langle n_{\text{ex}}(g, \beta, \mu) \rangle_{2\text{D}} = \infty$$

$$\lim_{\mu \uparrow -E_g a'_1} \langle n_{\text{ex}}(g, \beta, \mu) \rangle_{3\text{D}} = \infty$$

KREIN'S FORMULA

- **Disorder** potential in D spatial dimensions

$$H = H_0 + V_{\text{dis}}(\mathbf{r}) \qquad H_0 \equiv \frac{\mathbf{p}^2}{2m} + mgx$$

- Formal expansion of the Resolvent $w \in \mathbf{C}$

$$R(w) = [H - w]^{-1} = R_0(w) - \sum_{n=1}^{\infty} [R_0(w) V_{\text{dis}}]^n R_0(w)$$

- Green's function of the *impurity free* Resolvent $D > 1$

$$G_0^{(D)}(w, g; \mathbf{x}, \mathbf{x}') \equiv \langle \mathbf{x} | R_0(w) | \mathbf{x}' \rangle = -\frac{\pi \kappa}{E_g} \int \frac{d^{D-1}k}{(2\pi \hbar)^{D-1}}$$

$$\exp \left\{ \frac{i}{\hbar} \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') \right\} \frac{u[\xi(x_{<})] v[\xi(x_{>})]}{\text{Ai}'[\xi(0)]}$$

$$\Im m z > 0 \qquad \mathbf{x} \neq \mathbf{x}' \qquad x_{<}(x_{>}) = \min(\max)\{x, x'\}$$

$$\xi \equiv \kappa x + \xi_0 \qquad \xi_0 \equiv E_g^{-1} \left(\frac{\mathbf{k}^2}{2m} - w \right)$$

$$u(\xi) \equiv \text{Bi}'(\xi_0) \text{Ai}(\xi) - \text{Ai}'(\xi_0) \text{Bi}(\xi) \qquad v(\xi) \equiv \text{Ai}(\xi)$$

- **Point-like impurity** $V_{\text{dis}}(\mathbf{r}) = \lambda_D \delta^{(D)}(\mathbf{r})$

$$G^{(D)}(w, g, \lambda_D; \mathbf{x}, \mathbf{x}') =$$

$$G_0^{(D)}(w, g; \mathbf{x}, \mathbf{x}') - \frac{G_0^{(D)}(w, g; \mathbf{x}, \mathbf{0})G_0^{(D)}(w, g; \mathbf{0}, \mathbf{x}')}{(1/\lambda_D) + G_0^{(D)}(w, g; \mathbf{0}, \mathbf{0})}$$

$$\Im m z > 0 \quad \mathbf{x} \neq \mathbf{x}'$$

- As it stands the **denominator is meaningless** owing to **UV divergences at coincident points** of the *impurity free Resolvent* \Rightarrow **Renormalization / Subtraction**

- *Impurity free* Green's function at $\mathbf{x} = \mathbf{x}' = \mathbf{0}$

$$G_0^{(D)}(\zeta, g; \mathbf{0}, \mathbf{0}) = -C_D \int_0^\infty dy y^{(D-3)/2} \frac{\text{Ai}(y - \zeta)}{\text{Ai}'(y - \zeta)}$$

$$C_D \equiv \frac{\kappa^D (4\pi)^{(1-D)/2}}{E_g \Gamma[(D-1)/2]} \quad \zeta \equiv \frac{w}{E_g}$$

$$1 < \Re e D < 2$$

- Asymptotic behavior for large x

$$\frac{\text{Ai}(x)}{\text{Ai}'(x)} \sim x^{-1/2} \left[1 + O(x^{-3/2}) \right]$$

- **Renormalized Green's function at $\mathbf{x} = \mathbf{x}' = 0$**

$$G_{0,R}^{(D)}(\zeta, g, \eta; \mathbf{0}, \mathbf{0}) =$$

$$- C_D \int_0^\infty dy y^{(D-3)/2} \left[\frac{\text{Ai}(y - \zeta)}{\text{Ai}'(y - \zeta)} + (y + \eta)^{-1/2} \right]$$

$\eta > 0$ arbitrary $\Re D > 1$

- **Renormalized running coupling parameter**

$$[\lambda_D^R(\eta)]^{-1} = \lim_{\Lambda \rightarrow \infty} \left[\frac{1}{\lambda_D} - I_D(\Lambda, \eta) \right]$$

$$I_D(\Lambda, \eta) \equiv -C_D \int_0^\Lambda dy y^{(D-3)/2} (y + \eta)^{-1/2}$$

\Updownarrow

**Renormalized denominator
of the Krein's formula**

$$\frac{1}{\lambda_D} + G_0^{(D)}(\zeta, g; \mathbf{0}, \mathbf{0}) \equiv \frac{1}{\lambda_D^R} + G_{0,R}^{(D)}(\zeta, g, \eta; \mathbf{0}, \mathbf{0})$$

has one simple zero $\zeta_0 \equiv (E_0/E_g)$

in the interval $-\infty < \zeta_0 < -a'_1$

Singularities of the Resolvent

- ★ **cut** along the real positive energy axis $\Re w > -a'_1 E_g$

continuous spectrum

- ★ **simple pole** on the real energy axis $\Re w < -a'_1 E_g$

$$w_0 = E_0 = \zeta_0 E_g \quad \forall \lambda_D^R \in \mathbf{R}$$

one bound state always exists

- ★ **Complex poles** of the Resolvent $\forall \lambda_D^R \in \mathbf{R}$

resonances i.e. metastable states

the bound state energy E_0 labels

the one-parameter family of Self-Adjoint Extensions

of the symmetric Hamiltonian operator $H_0^{(D)}(g)$

- **Thermodynamic limit**

$$V_{\perp}, \langle N \rangle \rightarrow \infty \quad \langle n \rangle_D \equiv \frac{\langle N \rangle}{V_{\perp}} \quad \text{fixed}$$

- **The range of chemical potential when $\lambda_D^R \neq \pm\infty$**

$$-\infty < \mu \leq E_0 < -E_g a'_1$$

- **The mean density of particles in the excited states**

$$\langle n_{\text{ex}} \rangle_D = \lambda_T^{1-D} \sum_{j=1}^{\infty} g_{(D-1)/2} (\exp \{ \beta E_g a'_j + \beta \mu \}) < \infty$$

$$D = 2, 3 \quad \forall \mu \leq E_0 < -E_g a'_1$$

⇕

BEC always occurs in $D = 2, 3$ iff $\lambda_D^R \neq \pm\infty$

★ **Critical quantities** ★

$$\lambda_{T_c}^{D-1} \langle n \rangle_D \approx 4\pi (\kappa \lambda_{T_c})^{-3} g_{(D+2)/2} (e^{\beta_c E_0})$$

4. CONCLUSIONS

- Role of Disorder and External Fields in the remarkable Physical Effects for Ideal Systems
- Exactly Solvable Models allow to grasp the key Dynamical Mechanisms leading to those Effects
- Point-like Interaction in QM is related to the Self Adjoint Extensions of the Symmetric Hamiltonian
- Contact Interaction fully corresponds to the δ -like Interaction only in $D = 1$
- Contact Interaction always provide Bose-Einstein condensation but for the case $D = 2$ $\alpha \neq 0$ $E_0 > 0$
- Bose-Einstein condensation never occurs with the Uniform Field and without Disorder in $D = 2, 3$
- Bose-Einstein condensation always occurs with the Uniform Field and Disorder $\forall D = 1, 2, 3$
- Critical Quantities have been explicitly computed