TIME EVOLUTION IN QM WITH GRAVITY

• Hamiltonian: $\mathbf{r} = (x_1, x_2, x_3)$, $\mathbf{p} = (p_1, p_2, p_3)$

$$H_0 = \frac{\mathbf{p}^2}{2m} + mgx_3 \ , \tag{1}$$

• Natural units: quantum gravitational length and energy

$$\begin{aligned} \lambda_g \equiv \kappa^{-1} &= \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3} \ , \\ E_g \equiv \frac{mg}{\kappa} &= \frac{\hbar^2 \kappa^2}{2m} \ . \end{aligned} \tag{2}$$

• Eigenvalues and eigenfunctions [$\mathrm{Ai}(z)$ is the Airy's function]

$$\begin{aligned} &\Psi_{E,p_1,p_2}(x_1, x_2, x_3) \equiv \langle x_1, x_2, x_3 | E, p_1, p_2 \rangle = \\ &\sqrt{\frac{\kappa}{E_g}} (2\pi\hbar)^{-1} \exp\left\{ \frac{i}{\hbar}(p_1 x_1 + p_2 x_2 \right\} \mathrm{Ai}\left(\kappa x_3 + \frac{p_1^2 + p_2^2}{\hbar^2 \kappa^2} - \frac{E}{E_g} \right) \ , \end{aligned} \tag{3}$$

$$E \in \mathbf{R} \ , \qquad (p_1, p_2) \in \mathbf{R}^2 \ . \tag{4}$$

The spectrum is therefore purely continuous $-\infty < E < +\infty$ and the degeneracy is also doubly continuous and given by the transverse momenta.

$$\langle E', p_1', p_2' | E, p_1, p_2 \rangle = \delta(E - E')\delta(p_1 - p_1')\delta(p_2 - p_2') \ . \tag{5}$$

• Propagating Kernel or Time Evolution Green's Function

$$
\begin{aligned}
G(\mathbf{r},\mathbf{r}';t) &\equiv \left\langle \mathbf{r} \left| \exp\left\{-\frac{i}{\hbar}Ht\right\} \right| \mathbf{r}' \right\rangle \\
&= \int_{-\infty}^{+\infty} dE\ \exp\left\{-\frac{i}{\hbar}Et\right\} \int_{-\infty}^{+\infty} dp_1 \int_{-\infty}^{+\infty} dp_2\ \Psi_{E,p_1,p_2}(\mathbf{r})\Psi^*_{E,p_1,p_2}(\mathbf{r}') \\
&= \frac{\kappa}{E_g}\int_{-\infty}^{+\infty}\frac{dp_1}{h}\int_{-\infty}^{+\infty}\frac{dp_2}{h}\ \exp\left\{\frac{i}{\hbar}\left[p_1(x_1-x_1')+p_2(x_2-x_2')-\frac{p_1^2+p_2^2}{2m}t\right]\right\} \\
&\times \int_{-\infty}^{+\infty} dE\ \exp\left\{-\frac{i}{\hbar}Et\right\}\mathrm{Ai}\left(\kappa x_3-\frac{E}{E_g}\right)\mathrm{Ai}\left(\kappa x_3'-\frac{E}{E_g}\right)\ .
\end{aligned} \tag{6}
$$

• Gaussian integration over transverse momenta

$$
\begin{aligned}
G(\mathbf{r},\mathbf{r}';t) &= \frac{m\kappa}{ithE_g}\exp\left\{\frac{im}{2\hbar t}\left[(x_1-x_1')^2+(x_2-x_2')^2\right]\right\} \\
&\times \int_{-\infty}^{+\infty} dE\ \exp\left\{-\frac{i}{\hbar}Et\right\}\mathrm{Ai}\left(\kappa x_3-\frac{E}{E_g}\right)\mathrm{Ai}\left(\kappa x_3'-\frac{E}{E_g}\right)\ .
\end{aligned} \tag{7}
$$

• Integral representation of the Airy's function

$$
\mathrm{Ai}(z) = \int_{-\infty}^{+\infty}\frac{d\alpha}{2\pi}\ \exp\left\{i\alpha z+\frac{i}{3}\alpha^3\right\}\ . \tag{8}
$$

• Taking eq. (8) into account, a tedious calculation leads to

$$
\begin{aligned}
G(\mathbf{r},\mathbf{r}';t) &= \left(\frac{m}{2\pi i\hbar t}\right)^{3/2}\exp\left\{\frac{im}{2\hbar t}\left[(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2\right]\right\} \\
&\times \exp\left\{-\frac{i}{12}\left(\frac{t}{\hbar}E_g\right)^3-\frac{it}{2\hbar}mg(x_3+x_3')\right\}\ ;
\end{aligned} \tag{9}
$$

Notice that the very last factor of the above expression breaks the manifest translation invariance and that, in the limit $g \to 0$, one easily recovers the standard Propagating Kernel of the free particle, which corresponds to the first line of the RHS of eq. (9).

- Gaussian wave packet

$$\psi_0(\mathbf{r}) = (2\pi\sigma)^{-3/4} \exp\left\{-\frac{\mathbf{r}^2}{4\sigma}\right\} , \qquad \langle\psi_0|\psi_0\rangle = 1 . \tag{10}$$

- Time evolution of the probability distribution

$$\psi_0(\mathbf{r}, t) = \int d^3\mathbf{r}' \; G(\mathbf{r}, \mathbf{r}'; t) \; \psi_0(\mathbf{r}') . \tag{11}$$

Again, a tedious calculation leads to the result $[\ \mathbf{r} = (x, y, z)\ ]$

$$|\psi_0(\mathbf{r}, t)|^2 = [2\pi\sigma(t)]^{-3/2} \; \exp\left\{-\frac{x^2 + y^2 + [z(t)]^2}{2\sigma(t)}\right\} , \tag{12}$$

$$\sigma(t) \equiv \sigma \left[1 + \left(\frac{\hbar t}{2m\sigma}\right)^2\right] , \qquad z(t) \equiv z + \frac{1}{2}gt^2 . \tag{13}$$

- Position indeterminacy

$$\Delta^2 x(t) = \Delta^2 y(t) = \sigma(t) \; ; \tag{15a}$$

$$\Delta^2 z(t) = \sigma(t) \left[1 + \frac{g^2 t^4}{4\sigma(t)}\right] . \tag{15b}$$

• Heat Kernel: $[\ t \longmapsto -i\hbar\beta\ ]$

$$
\begin{aligned}
&\langle \mathbf{r}\,|\exp\{-\beta H\}|\,\mathbf{r}'\rangle \equiv G(\mathbf{r},\mathbf{r}';t=-i\hbar\beta)\\
&= \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\frac{1}{12}\left(\beta E_g\right)^3 - \beta m g(x_3+x_3')\right\}\\
&\times \exp\left\{-\frac{m}{2\hbar^2\beta}\left[(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2\right]\right\}\ ;
\end{aligned}
\tag{16}
$$

• Diagonal Heat Kernel

$$
\langle \mathbf{r}\,|\exp\{-\beta H\}|\,\mathbf{r}\rangle = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\frac{1}{12}\left(\beta E_g\right)^3 - 2\beta m g x_3\right\}\ . \tag{17}
$$

• Density of the states

$$
\rho^{(3D)}(E,g) = \frac{m\kappa}{2\pi\hbar^2}\left[\frac{E}{E_g}\mathrm{Ai}^2\left(-\frac{E}{E_g}\right) + \mathrm{Ai}'^2\left(-\frac{E}{E_g}\right)\right]\ . \tag{18}
$$

• Spectral property

$$
\begin{aligned}
&\exp\{2\beta m g x_3\}\,\langle \mathbf{r}\,|\exp\{-\beta H\}|\,\mathbf{r}\rangle =\\
&\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\frac{1}{12}\left(\beta E_g\right)^3\right\} = \int_{-\infty}^{+\infty} dE\ \rho^{(3D)}(-E,g)\exp\{\beta E\}\ .
\end{aligned}
\tag{19}
$$