KREIN'S FORMULA

1. Two dimensional kinetic Hamiltonian

The $O(2)$ invariant $2D$ kinetic operator admits a one-parameter family of self-adjoint extensions (SAE), which can be labelled by some energy parameter $E_0 > 0$. The spectral decomposition of a generic element within the family reads

$$H(E_0) = \sum_{l=-\infty}^{+\infty} \int_0^\infty dk \frac{\hbar^2 k^2}{2m} |l, k\rangle \langle k, l| - E_0 |\psi_B\rangle \langle\psi_B| \ , \tag{1}$$

where

$$\langle r, \theta | l, k\rangle = \frac{\exp\{il\theta\}}{\sqrt{2\pi}} \psi_l(k, r; E_0) \ , \qquad k \geq 0 \ , \tag{2}$$

in which, specifically,

$$\psi_l(k, r) = \sqrt{k} J_l(kr) \ , \qquad l \in \mathbf{Z} - \{0\} \ , \tag{3}$$

whereas

$$\psi_0(k, r; E_0) = A(k; E_0) J_0(kr) + B(k; E_0) N_0(kr) \ . \tag{4}$$

The coefficients $A(k; E_0)$ and $B(k; E_0)$ are such that

$$\frac{B(k; E_0)}{A(k; E_0)} = \frac{(-\pi)}{\ln\left(\hbar^2 k^2 / 2mE_0\right)} \ . \tag{5}$$

The normalizable bound state is provided by

$$\langle r|\psi_B\rangle = \psi_B(\kappa, r) = \frac{\kappa}{\sqrt{\pi}} K_0(\kappa r) \ , \qquad \hbar\kappa \equiv \sqrt{2mE_0} \ . \tag{6}$$

Notice that the eigenstates are normalized according to

$$\langle\psi_B|\psi_B\rangle = 1 \ , \qquad \langle l', k'|l, k\rangle = \delta_{l,l'} \delta(k - k') \ , \quad k, k' > 0 \ .$$

Now we aim to show that the very same conclusion is reached following the Krein's method for the resolvent. Let us begin from the resolvent of the free kinetic operator in $2D$ which is defined by

$$(H_0 - z) G_0(\mathbf{r}, \mathbf{r}'; z) = \delta^{(2)}(\mathbf{r} - \mathbf{r}') \ , \tag{7}$$

where

$$H_0 = \frac{\mathbf{p}^2}{2m} \ .$$

The resolvent can be written in terms of the basic integral representation

$$
\begin{aligned}
G_0(\mathbf{r}-\mathbf{r}';z) &\equiv \int \frac{d^2p}{h^2} \frac{\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')\right]}{\frac{\mathbf{p}^2}{2m}-z} \\
&= \frac{2\pi m}{h^2}\int_0^\infty \frac{dw}{w-z}\; J_0\left(\frac{\sqrt{2mw}}{\hbar}|\mathbf{r}-\mathbf{r}'|\right) \\
&= \frac{im}{2\hbar^2} H_0^{(1)}(q|\mathbf{r}-\mathbf{r}'|)\ , \\
\hbar q &\equiv \sqrt{2mz}\ , \qquad \Im\mathrm{m} z > 0.
\end{aligned} \tag{8}
$$

It is worthwhile to remark that if we set $z = E + i\eta$ we obtain in the limit $\eta \downarrow 0$ and $E > 0$

$$
\begin{aligned}
G_0(\mathbf{r}-\mathbf{r}';z) = &\frac{2\pi m}{h^2}\int_0^\infty dw\ \mathrm{CPV}\left(\frac{1}{w-E}\right)\ J_0\left(\frac{\sqrt{2mw}}{\hbar}|\mathbf{r}-\mathbf{r}'|\right) \\
&+ i\frac{2\pi^2 m}{h^2} J_0\left(\frac{\sqrt{2mE}}{\hbar}|\mathbf{r}-\mathbf{r}'|\right)\ .
\end{aligned}
$$

From the above expression it immediately follows that

$$
\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\left\{G_0(\mathbf{r},\mathbf{r};E+i\eta) - G_0(\mathbf{r},\mathbf{r};E-i\eta)\right\} = \frac{1}{\pi}\Im\mathrm{m}\ G_0(\mathbf{r},\mathbf{r};E) = \frac{2\pi m}{h^2} \equiv \rho_0^{(2D)}\ ,
$$

as it does owing to translation invariance, where $\rho_0^{(2D)}$ is the density of the quantum states *per* unit area in two space dimensions.

Notice that

$$
\begin{aligned}
H_0^{(1)}(q|\mathbf{r}-\mathbf{r}'|) &= J_0(q|\mathbf{r}-\mathbf{r}'|) + iN_0(q|\mathbf{r}-\mathbf{r}'|) \\
&\sim 1 + \frac{2i}{\pi}\left[\ln(q|\mathbf{r}-\mathbf{r}'|) - \ln 2 + \gamma_E\right]\ , \qquad |\mathbf{r}-\mathbf{r}'| \sim 0\ .
\end{aligned} \tag{9}
$$

It follows therefrom that

$$
\begin{aligned}
&G_0(\mathbf{r}-\mathbf{r}';z) \sim -\frac{m}{2\pi\hbar^2}\times \\
&\left\{\ln\left(\frac{q}{q_0}\right)^2 + \ln\left(\frac{q_0|\mathbf{r}-\mathbf{r}'|}{2}\right)^2 + 2\gamma_E - i\pi\right\}\ , \\
&|\mathbf{r}-\mathbf{r}'| \sim 0\ ,
\end{aligned} \tag{10}
$$

where $\hbar q_0$ is some arbitrary momentum scale. It is important to realize that the resolvent (10) exhibits an ultraviolet divergence at coincident points $\mathbf{r} = \mathbf{r}'$. To this concern it is convenient to introduce the renormalized (or subtracted) resolvent: namely,

$$
G_0^R(\mathbf{r}-\mathbf{r}';z) = G_0(\mathbf{r},\mathbf{r}';z) + \frac{m}{\pi\hbar^2}\ln\left(\frac{q_0|\mathbf{r}-\mathbf{r}'|}{2}\right)\ . \tag{11}
$$

The renormalized resolvent turns out to be finite, although arbitrary, at coincident points as we have

$$G_0^R(\mathbf{0},\mathbf{0};z) = \frac{m}{2\pi\hbar^2}\left\{ i\pi - \ln\left(\frac{q}{q_0}\right)^2 - 2\gamma_E \right\} \ , \tag{12}$$

up to the arbitrariness in the choice of the renormalization prescription, *i.e.* the subtraction momentum scale $\hbar q_0$.

Now we aim to obtain the resolvent in the presence of contact interaction. To this purpose, following [Alb], let us start from the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) \tag{13}$$

and expand the resolvent

$$G(z) = \left[\frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) - z\right]^{-1} = G_0(z) - \sum_{n=1}^{\infty}[G_0(z)V]^n G_0(z) \ . \tag{14}$$

If we now formally insert $V(\mathbf{r}) = \lambda\delta^{(2)}(\mathbf{r})$ and consider the integral kernel, *i.e.* the Green's function, we formally obtain

$$\begin{aligned} G(\mathbf{r},\mathbf{r}';z) &= G_0(q|\mathbf{r}-\mathbf{r}'|) - \frac{G_0(q|\mathbf{r}|)G_0(q|\mathbf{r}'|)}{\frac{1}{\lambda} + G_0(\mathbf{0},\mathbf{0};z)} \ , \\ \hbar q &\equiv \sqrt{2mz} \ , \qquad \Im\mathrm{m} z > 0. \end{aligned} \tag{15}$$

In order to make the denominator meaningful, we have to implement the renormalization (or subtraction) procedure, which amounts to the replacement

$$\begin{aligned} &\frac{1}{\lambda} + G_0(\mathbf{0},\mathbf{0};z) \equiv \frac{1}{\lambda_R} + G_0^R(\mathbf{0},\mathbf{0};z) = \\ &\frac{1}{\lambda_R} + \frac{m}{2\pi\hbar^2}\left\{ i\pi - \ln\left(\frac{q}{q_0}\right)^2 - 2\gamma_E \right\} + \text{constant} \ , \end{aligned} \tag{16}$$

in which we have introduced the renormalized and scale dependent coupling parameter λ_R in order to keep finite the denominator of eq. (15). A comparison with eq.s (1)-(6) is mandatory to fix the renormalization prescrption, *i.e.* the arbitrary constant in eq. (16). To this aim, let us consider the Bergmann-Manuel-Tarrach (BMT) renormalization prescription [BMT], which is defined by

$$\left.\frac{1}{\lambda_R} + G_0^R(\mathbf{0},\mathbf{0};z)\right|_{\mathrm{BMT}} \equiv -\frac{m}{2\pi\hbar^2}\ln\left(-\frac{\hbar^2 q^2}{2mE_0}\right) \ . \tag{17}$$

In the RHS of the above expression the energy scale E_0 is nothing but the absolute value of the bound state energy, whereas the momentum scale $\hbar q_0 \geq 0$ is the subtraction point at which the "running" coupling parameter λ_R is defined: namely,

$$\lambda_R(q_0) = \frac{2\pi\hbar^2}{m\ln\left(\frac{\hbar^2 q_0^2}{2mE_0}\right)} \ . \tag{18}$$

It is worthwhile to observe that, from the field theoretical point of view, the behaviour of the "running" coupling parameter $\lambda_R(q_0)$, at fixed E_0, turns out to be somewhat peculiar: as a matter of fact, the model under investigation appears to be asymptotically free and infrared stable at the same time because

$$\lim_{q_0\to 0}\lambda_R(q_0)=0=\lim_{q_0\to\infty}\lambda_R(q_0)\ .$$

Now, taking the relationships (8),(16)-(18) into account, we can rewrite eq. (15) in the form

$$\begin{aligned} G(\mathbf{r},\mathbf{r}';z) &= \frac{im}{2\hbar^2}H_0^{(1)}(q|\mathbf{r}-\mathbf{r}'|)-\frac{\pi m}{2\hbar^2}\frac{H_0^{(1)}(qr)H_0^{(1)}(qr')}{\ln\left(-\frac{\hbar^2q^2}{2mE_0}\right)}\ , \\ \hbar q &\equiv \sqrt{2mz}\ ,\qquad \Im\mathrm{m}z>0. \end{aligned} \tag{19}$$

Let us first check that the singularity in the denominator of the above equation just corresponds to the presence of a bound state. As a matter of fact we have

$$\begin{aligned} -\lim_{z\to -E_0}(z+E_0)G(\mathbf{r},\mathbf{r}';z) &= \lim_{z\to -E_0}\frac{\pi m(z+E_0)}{2\hbar^2}\frac{H_0^{(1)}[q(z)r]H_0^{(1)}[q(z)r']}{\ln\left[1-\left(\frac{z}{E_0}+1\right)\right]} \\ &= \frac{2mE_0}{\pi\hbar^2}K_0(\kappa r)K_0(\kappa r')=\frac{\kappa^2}{\pi}K_0(\kappa r)K_0(\kappa r')\ , \\ \hbar\kappa &= \sqrt{2mE_0}\ ,\qquad 0<E_0\le\infty\ , \end{aligned} \tag{20}$$

in perfect agreement with eq. (6). The continuous part of the spectrum can be read off the imaginary part of the Green's function, taking the summation theorem for the Hankel's function into account [Gra] and after setting $\mathbf{r}=r\cos\theta,\ \mathbf{r}'=r'\cos\theta',\ \varphi=\theta-\theta'$: namely,

$$\begin{aligned} \frac{\hbar^2 q}{\pi m}\,\Im\mathrm{m}\ G\left(\mathbf{r},\mathbf{r}';z=\frac{\hbar^2q^2}{2m}\right) &= \sum_{l\in\mathbf{Z}-\{0\}}\frac{q}{2\pi}e^{il\varphi}J_l(qr)J_l(qr') \\ &+ \frac{(q/2\pi)}{\ln^2\left(\frac{\hbar^2q^2}{2mE_0}\right)+\pi^2}\left[J_0(qr)J_0(qr')\ln^2\left(\frac{\hbar^2q^2}{2mE_0}\right)+\pi^2N_0(qr)N_0(qr')\right] \\ &- \frac{q}{2}\frac{\ln\left(\frac{\hbar^2q^2}{2mE_0}\right)}{\ln^2\left(\frac{\hbar^2q^2}{2mE_0}\right)+\pi^2}\left[J_0(qr)N_0(qr')+N_0(qr)J_0(qr')\right]\ . \end{aligned} \tag{21}$$

It can be readily verified that if we set

$$\begin{aligned} \psi_0(q,r;E_0) &\equiv \sqrt{\frac{(q/2\pi)}{\ln^2\left(\frac{\hbar^2q^2}{2mE_0}\right)+\pi^2}}\left[\ln\left(\frac{\hbar^2q^2}{2mE_0}\right)J_0(qr)-\pi N_0(qr)\right] \\ &= \sqrt{q}\cos[\pi\mu(q;E_0)]J_0(qr)+\sqrt{q}\sin[\pi\mu(q;E_0)]N_0(qr)\ , \end{aligned} \tag{22}$$

eq. (21) can be recast into the form

$$\frac{\hbar^2 q}{\pi m}\ \Im\mathrm{m}\ G\left(\mathbf{r},\mathbf{r}';z=\frac{\hbar^2 q^2}{2m}\right)=\frac{q}{\pi}\sum_{l=1}^{\infty}J_l(qr)J_l(qr')\cos(l\varphi)+\psi_0(q,r;E_0)\psi_0(q,r';E_0)\ . \tag{23}$$

After direct inspection one can easily realize that eq. (23) does agree with the spectral decomposition of eq. (1), whilst eq. (22) does coincide with eq. (4) leading to the identification

$$\frac{B(q;E_0)}{A(q;E_0)}=\frac{(-\pi)}{\ln\left(\hbar^2 q^2/2mE_0\right)}=\tan[\pi\mu(q;E_0)]\ . \tag{24}$$

To sum up, we have explicitely verified that the method of boundary conditions, or von Neumann method of deficiency indices, to obtain all the SAE of the free $2D$ kinetic operator is equivalent to the Krein's formula for the resolvent of the hamiltonian operator involving contact interaction. The full correspondence between the two methods is achieved, provided some renormalization prescription is adopted which defines the running coupling parameter of contact interaction.

References

[Alb] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York (1988) 109-110, 357-358.

[BMT] O. Bergmann, Phys. Rev. D **46**, 5474 (1992);
C. Manuel and R. Tarrach, Phys. Lett. B **268**, 222 (1991).

[Gra] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego (1994) see **8.531**2. pg 993.

2. Two dimensional uniform field

Here we apply the Krein's formula to the case of a two dimensional point particle in the presence of a uniform (*i.e.* constant and homogeneous) field, such as gravity, and contact interaction. The first step is to find the resolvent of the Hamiltonian in the absence of contact interaction. The resolvent is defined by

$$(H_0 - z)G_0(\mathbf{r}, \mathbf{r}'; z) = \delta^{(2)}(\mathbf{r} - \mathbf{r}') \ , \tag{1}$$

where

$$H_0 = \frac{\mathbf{p}^2}{2m} + mgx_2 \ , \tag{2}$$

in which we have set $\mathbf{r} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$. Here the absence of contact interaction corresponds to assume regularity of the eigenfunctions on the whole plane, that means in turn $[H_0, p_1] = 0$. It is worthwhile to notice that the presence of a uniform field just allows to introduce natural quantum gravitational wavelength and energy: namely,

$$\begin{aligned} \lambda_g \equiv \kappa^{-1} &= \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3} \ , \\ E_g \equiv \frac{mg}{\kappa} &= \frac{\hbar^2\kappa^2}{2m} \ . \end{aligned} \tag{3}$$

It is convenient to introduce dimensionless coordinates and quantum numbers, in such a way that the eigenvalues equation for the Hamiltonian (2) becomes

$$\left(\partial_x^2 + \partial_y^2 - y + \epsilon\right)\psi_\epsilon(x, y) = 0 \ , \tag{4}$$

where

$$x \equiv \kappa x_1 \ , \quad y \equiv \kappa x_2 \ , \quad \epsilon \equiv \frac{E}{E_g} \ . \tag{5}$$

After setting

$$\psi_\epsilon(x, y) = e^{ipx}\psi_{\epsilon,p}(y) \ , \qquad p \equiv \frac{p_1}{\hbar\kappa} \ , \tag{6}$$

we eventually get

$$\psi''_{\epsilon,p} - \left(y + p^2 - \epsilon\right)\psi_{\epsilon,p} = 0 \ , \qquad \epsilon, p \in \mathbf{R} \ . \tag{7}$$

As gravity is attractive in the half plane $y > 0$, the only allowed eigenfunctions which are square summable in the half plane $y > 0$ are provided by

$$\psi_{\epsilon,p}(x, y) = (2\pi)^{-1/2} e^{ipx} \mathrm{Ai}\left(y + p^2 - \epsilon\right) \equiv \langle x, y|\epsilon, p\rangle \ , \qquad \epsilon, p \in \mathbf{R} \ , \tag{8}$$

where $\mathrm{Ai}(z)$ is the Airy's function [Abr]. The above dimensionless improper wave-functions are normalized according to

$$\langle \epsilon', p'|\epsilon, p\rangle = \delta(\epsilon - \epsilon')\delta(p - p') \ . \tag{9}$$

Going back to dimensionfull quantities the eigenfunctions read

$$\begin{aligned}\Psi_{E,p_1}(x_1,x_2) &\equiv \langle x_1, x_2 | E, p_1\rangle \\ &= \sqrt{\frac{\kappa}{2\pi\hbar E_g}}\exp\left\{\frac{i}{\hbar}p_1 x_1\right\}\mathrm{Ai}\left(\kappa x_2 + \frac{p_1^2}{\hbar^2\kappa^2} - \frac{E}{E_g}\right) \ ,\end{aligned} \tag{10}$$

which turn out to be normalized according to

$$\langle E', p_1' | E, p_1\rangle = \delta(E - E')\delta(p_1 - p_1') \ . \tag{11}$$

It is interesting to obtain the weak field limit $g \sim 0$ of the eigenfunctions (10). To this aim, if we set

$$u \equiv \frac{E}{E_g} - \frac{p_1^2}{\hbar^2\kappa^2} - \kappa x_2 \ , \qquad \zeta \equiv \frac{2}{3}u^{3/2} \ , \tag{12}$$

we easily find [Abr]

$$\begin{aligned}\Psi_{E,p_1}(x_1,x_2) &= \sqrt{\frac{\kappa}{2\pi\hbar E_g}}\exp\left\{\frac{i}{\hbar}p_1 x_1\right\}\mathrm{Ai}(-u) \\ &\sim \sqrt{\frac{\kappa}{2\pi^2\hbar E_g}}\exp\left\{\frac{i}{\hbar}p_1 x_1\right\}u^{-1/4}\sin\left(\zeta + \frac{\pi}{4}\right) \\ &= \sqrt{\frac{\kappa}{2\pi^2\hbar E_g}}\exp\left\{\frac{i}{\hbar}p_1 x_1\right\}\left(\frac{\mathcal{E}}{E_g}\right)^{-1/4}\left(1 + \frac{1}{4}\kappa x_2\frac{E_g}{\mathcal{E}} + \ldots\right)\sin\left(\zeta + \frac{\pi}{4}\right) \\ & g \sim 0 \ , \quad u > 0 \ , \quad \mathcal{E} \equiv E - \frac{p_1^2}{2m} \ .\end{aligned} \tag{13}$$

Now, taking into account that

$$E_g = g^{2/3}\left(\frac{m\hbar^2}{2}\right)^{1/3} \ ; \qquad \zeta \sim \frac{2}{3}\left(\frac{\mathcal{E}}{E_g}\right)^{3/2}\left(1 - \frac{3}{2}\kappa x_2\frac{E_g}{\mathcal{E}} + \ldots\right) \ , \quad g \sim 0 \ , \tag{14}$$

we eventually obtain

$$\begin{aligned}&\Psi_{E,p_1}(x_1,x_2) \sim \frac{1}{\pi\hbar}\left(\frac{m}{2\mathcal{E}}\right)^{1/4}\exp\left\{\frac{i}{\hbar}p_1 x_1\right\}\sin\left[\frac{\pi}{4} + \frac{2}{3}\left(\frac{\mathcal{E}}{E_g}\right)^{3/2} - \frac{1}{\hbar}p_2 x_2\right] \ , \\ & g \sim 0 \ , \quad E > \frac{p_1^2}{2m} + mgx_2 \ , \quad p_2 \equiv \sqrt{2m\mathcal{E}} \ .\end{aligned} \tag{15}$$

Notice that the asymptotic form (15) of each vawe-functions just corresponds to the sum of two opposite progressive plane waves in the x_2 direction, up to a phase factor [Lan], whilst the eigenfunctions exponentially vanish when $u < 0$, as it does in the case of negative energies.

Let us come to the evaluation of the Green's function wich is defined to be

$$\begin{aligned} G_0(\mathbf{r},\mathbf{r}';z) &\equiv \langle \mathbf{r}|(H_0-z)^{-1}|\mathbf{r}'\rangle \\ &= \int_{-\infty}^{+\infty}\frac{dw}{w-z}\int_{-\infty}^{+\infty} dp_1\ \Psi_{w,p_1}(x_1,x_2)\Psi^*_{w,p_1}(x_1',x_2') \\ &= \frac{\kappa}{2\pi\hbar E_g}\int_{-\infty}^{+\infty} dp_1\ \exp\left\{\frac{i}{\hbar}p_1(x_1-x_1')\right\} \\ &\times \int_{-\infty}^{+\infty}\frac{dw}{w+\frac{p_1^2}{2m}-z}\mathrm{Ai}\left(y-\frac{w}{E_g}\right)\mathrm{Ai}\left(y'-\frac{w}{E_g}\right)\ . \end{aligned} \tag{16}$$

The above expression can be rewritten in terms of the following integral representation: namely,

$$\begin{aligned} G_0(\mathbf{r},\mathbf{r}';z) &= \lim_{\varepsilon\downarrow 0}\frac{im}{2\pi\hbar^2}\int_0^\infty \frac{dt}{\varepsilon+it}\exp\left\{-\frac{(x-x')^2}{4(\varepsilon+it)}\right\} \\ &\times \exp\left\{it\frac{2mz}{\hbar^2}-i\frac{t^3}{3}-i\frac{yy'}{t}+\frac{i}{4t}(y+y'-t^2)^2\right\}\ ,\qquad \Im\mathrm{m} z>0\ . \end{aligned} \tag{17}$$

Notice that the limit has been introduced in order to carefully perform the integration over the degeneracy variable p_1 in eq. (16) and to treat correctly the case of coincident points. As a matter of fact, it is not difficult to realize that the limits $\mathbf{r}\to\mathbf{r}'$ and $\varepsilon\downarrow 0$ do not commute. After some straightforward algebra we obtain

$$\begin{aligned} &G_0(\mathbf{r},\mathbf{r}';z) = \frac{m}{2\pi\hbar^2}\int_0^\infty \frac{dt}{t}\exp\left\{it\left[\frac{2mz}{\hbar^2}-\frac{\kappa^3}{2}(x_2+x_2')\right]-\frac{i}{12}t^3\kappa^6+\frac{i}{4t}(\mathbf{r}-\mathbf{r}')^2\right\}\ , \\ &\Im\mathrm{m} z>0\ ,\qquad \mathbf{r}\neq\mathbf{r}'\ . \end{aligned} \tag{18}$$

Now some comments are in order. First we remark that one should expect that translation invariance holds true, as we are in the presence of a uniform field and in the absence of contact interaction. Translation invariance becomes manifest if we turn to the new complex energy variable

$$\xi \equiv z-\frac{\hbar^2\kappa^3}{4m}(x_2+x_2')\ ,\qquad \Im\mathrm{m}\xi>0\ , \tag{19}$$

in such a way that we can employ the manifestly translation invariant form of the Green's function, *i.e.*

$$\begin{aligned} &G_0(\mathbf{r}-\mathbf{r}';\xi) = \frac{m}{2\pi\hbar^2}\int_0^\infty \frac{dt}{t}\exp\left\{it\frac{2m\xi}{\hbar^2}-\frac{i}{12}t^3\kappa^6+\frac{i}{4t}(\mathbf{r}-\mathbf{r}')^2\right\}\ , \\ &\Im\mathrm{m}\xi>0\ ,\qquad \mathbf{r}\neq\mathbf{r}'\ . \end{aligned} \tag{20}$$

The second remark concerns the limit of vanishing uniform field. It can be readily checked that

$$\begin{aligned} \lim_{g\downarrow 0} G_0(\mathbf{r}-\mathbf{r}';\xi) &= \frac{m}{2\pi\hbar^2}\int_0^\infty \frac{dt}{t}\exp\left\{it\frac{2mz}{\hbar^2}+\frac{i}{4t}(\mathbf{r}-\mathbf{r}')^2\right\} \\ &= \frac{im}{2\hbar^2}H_0^{(1)}\left(\frac{\sqrt{2mz}}{\hbar}|\mathbf{r}-\mathbf{r}'|\right)\ ,\qquad \Im\mathrm{m} z>0\ , \end{aligned} \tag{21}$$

in agreement with eq. (8) of section 1. Thanks to translation invariance and taking eq. (17) properly into account, we can also extract the density of the quantum states *per* unit volume: namely,

$$
\begin{aligned}
\rho_g^{(2D)}(E) &\equiv \lim_{\eta\downarrow 0} \frac{1}{2\pi i}\left\{G_0(\mathbf{0},\mathbf{0};z=E+i\eta)-G_0(\mathbf{0},\mathbf{0};z=E-i\eta)\right\} \\
&\equiv \frac{1}{\pi}\Im\mathrm{m}\ G_0(\mathbf{0},\mathbf{0};z=E) \\
&= \lim_{\varepsilon\downarrow 0}\frac{2m}{\hbar^2}\int_0^\infty \frac{dt}{t^2+\varepsilon^2}\left\{\varepsilon\cos\left(\frac{2mE}{\hbar^2}t-\frac{\kappa^6}{12}t^3\right)+t\sin\left(\frac{2mE}{\hbar^2}t-\frac{\kappa^6}{12}t^3\right)\right\} \\
&= \frac{2m}{\hbar^2}\left\{\frac{\pi}{2}+\int_0^\infty\frac{dt}{t}\sin\left(\frac{2mE}{\hbar^2}t-\frac{\kappa^6}{12}t^3\right)\right\}\ , \qquad E\in\mathbf{R}\ .
\end{aligned}
\tag{22}
$$

As a check, we observe that

$$
\lim_{g\downarrow 0}\rho_g^{(2D)}(E)=\vartheta(E)\frac{2m\pi}{h^2}=\rho_0^{(2D)}\ , \tag{23}
$$

where ϑ is the Heaviside's step distribution.

Now we are ready to obtain the renormalized Green's function $G_0^R(\mathbf{r}-\mathbf{r}';\xi)$. To this aim it is convenient to rewrite eq. (20) in the form

$$
\begin{aligned}
G_0(\mathbf{r}-\mathbf{r}';\xi) &= \frac{im}{2\hbar^2}H_0^{(1)}\left(\frac{\sqrt{2m\xi}}{\hbar}|\mathbf{r}-\mathbf{r}'|\right) \\
&+\frac{m}{2\pi\hbar^2}\int_0^\infty\frac{dt}{t}\exp\left\{it\frac{2m\xi}{\hbar^2}+\frac{i}{4t}(\mathbf{r}-\mathbf{r}')^2\right\}\left(\exp\left\{-it^3\frac{\kappa^6}{12}\right\}-1\right)\ , \\
&\Im\mathrm{m}\xi>0\ ,\qquad \mathbf{r}\neq\mathbf{r}'\ .
\end{aligned}
\tag{24}
$$

As a consequence, it is clear that the ultraviolet divergence at coincident points is subtracted by the very same term as in the absence of gravity: namely,

$$
G_0^R(\mathbf{r}-\mathbf{r}';\xi)=G_0(\mathbf{r}-\mathbf{r}';\xi)+\frac{m}{\pi\hbar^2}\ln\left(\frac{q_0|\mathbf{r}-\mathbf{r}'|}{2}\right)\ . \tag{25}
$$

according to eq. (11) of section 1, where $\hbar q_0$ is some arbitrary momentum scale. Again, the renormalized Green's function turns out to be finite, although arbitrary, at coincident points as we have

$$
G_0^R(\mathbf{0},\mathbf{0};z)=\frac{m}{2\pi\hbar^2}\left\{i\pi-\ln\left(\frac{2mz}{\hbar^2q_0^2}\right)-2\gamma_E+I\left(z,g\right)\right\}\ , \tag{26}
$$

where

$$
I(z,g)\equiv\int_0^\infty\frac{dt}{t}e^{itz}\left(\exp\left\{-img^2\hbar^2\frac{t^3}{24}\right\}-1\right)\ . \tag{27}
$$

In order to introduce contact interaction and obtain the corresponding Krein's formula for the Green's function - see eq. (15) of section 1 - we have to define the renormalized coupling of the contact interaction. This can be done in close analogy with eq. (16) of section 1: namely,

$$\begin{aligned}&\frac{1}{\lambda}+G_0(\mathbf{0},\mathbf{0};z)\equiv\frac{1}{\lambda_R}+G_0^R(\mathbf{0},\mathbf{0};z)=\\&\frac{1}{\lambda_R}+\frac{m}{2\pi\hbar^2}\left\{i\pi-\ln\left(\frac{2mz}{\hbar^2q_0^2}\right)-2\gamma_E+I(z,g)\right\}+\text{constant}\ ,\end{aligned}\tag{28}$$

in which, again, we have introduced the renormalized and scale dependent coupling parameter λ_R in order to keep finite the denominator of the Krein's formula. Keeping as well the BMT renormalization prescription, eq. (28) becomes equivalent to the following pair of equations

$$\left.\frac{1}{\lambda_R}+G_0^R(\mathbf{0},\mathbf{0};z)\right|_{\text{BMT}}\equiv-\frac{m}{2\pi\hbar^2}\left\{\ln\left(-\frac{z}{E_0}\right)-I(z,g)\right\}\ ,\tag{29a}$$

$$\lambda_R(q_0)=\frac{2\pi\hbar^2}{m\ln\left(\frac{\hbar^2q_0^2}{2mE_0}\right)}\ ,\qquad E_0>0\ .\tag{29b}$$

In the RHS of the above expression the energy scale E_0 is nothing but the absolute value of the bound state energy in the absence of gravity, whereas the momentum scale $\hbar q_0\geq 0$ is the subtraction point at which the "running" coupling parameter λ_R is defined. It is apparent from eq. (29a) that there are no poles on the real axis of the energy variable z as long as $g\neq 0$. As a matter of fact, the basic quantity $I(z,g)$ always contains an imaginary part when $\Im\text{m}z=0,\ g\neq 0$. On the other hand, the bound state arises in the limit $g=0$ as $I(z,g=0)=0$ - see eq. (27) - according to eq. (20) of section 1. This means that in the presence of gravity and contact interaction the spectrum is purely continuous and coincident with the whole real energy axis. Consequently, once gravity is switched on, no matter how weak it is, the "unperturbed" bound state, due to pure contact interaction, becomes a metastable state whose decay width will be evaluated below.

Now we are ready to obtain the Krein's formula in the presence of gravity. From the expression

$$\begin{aligned}&G_0(\mathbf{r};z)=\frac{m}{2\pi\hbar^2}\int_0^\infty\frac{dt}{t}\exp\left\{itz-it\frac{mg}{2}x_2-\frac{i}{24}mg^2\hbar^2t^3+i\frac{m\mathbf{r}^2}{2\hbar^2t}\right\}\ ,\\&\Im\text{m}z>0\ ,\qquad \mathbf{r}\neq\mathbf{0}\ ,\end{aligned}\tag{30}$$

we obtain the Krein's formula for the Green's function in the presence of gravity and contact interaction: namely,

$$\begin{aligned}&G(\mathbf{r},\mathbf{r}';z)=G_0(\mathbf{r},\mathbf{r}';z)+\frac{2\pi\hbar^2}{m}\frac{G_0(\mathbf{r};z)G_0(\mathbf{r}';z)}{\ln\left(-\frac{z}{E_0}\right)-I(z,g)}\ ,\\&\Im\text{m}z>0\ .\end{aligned}\tag{31}$$

The above exact expression for the Green's function exhibits the loss of translation invariance along the x_1-direction. This means that the translation operator p_1 does not commute with the Hamiltonian. As a consequence, we can no longer use the eigenvalues of p_1 to label the degeneracy of the eigenstates of the self-adjoint Hamiltonians $H(E_0)$, whose Green's functions are provided by eq. (31).

References

[Abr] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972) 446-452.

[Lan] L. D. Landau and E. M. Lifsits, *Meccanica quantistica - Teoria non relativistica*, Editori Riuniti, Roma (1976) p. 102.