EXACT SOLUTION OF THE ONE-IMPURITY QUANTUM HALL PROBLEM

Paola Giacconi [a] and **Roberto Soldati** [b]

Dipartimento di Fisica "A. Righi", Università di Bologna,

Istituto Nazionale di Fisica Nucleare, Sezione di Bologna,

40126 Bologna - Italy

Abstract

The problem of a non-relativistic electron in the presence of a uniform electromagnetic field and of one impurity, described by means of an Aharonov-Bohm point-like vortex, is studied. The exact solution is found, which corresponds to the diagonalization of the essentially self-adjoint quantum Hamiltonian. The quantum Hall conductance is computed and shown to be the same as in the impurity-free case. This exactly solvable model seems to give indications, concerning the possible microscopic mechanisms underlying the integer quantum Hall effect, which sensibly deviate from some proposals available in the literature.

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[a] e-mail: giacconi@bo.infn.it

[b] e-mail: soldati@bo.infn.it

1. Introduction.

The discovery of the integer quantum Hall effect [1] represents one of the most remarkable experimental finding in the last years. The effort towards an explanation of the experimental plots for the Hall conductance vs electron density or applied magnetic field has stimulated a huge theoretical activity [2]. The common key ingredients, among theoretical models that have been put forward, are the irrelevance of electrons interactions and the central role played by the presence of impurities within the Hall sample, *i.e.*, the effect of disorder. It appears therefrom, that the study of the *2+1* dimensional quantum dynamics of a non-relativistic electron, in the presence of background electromagnetic fields and of suitable potentials describing disorder, is of essential importance. Such a simple model should represent the natural starting point in order to achieve a microscopic description for the integer quantum Hall phenomenology.

The simplest way to draw some localized impurity could appear to be a point-like interaction as described by a δ-like potential [3]. However, it turns out that quantum mechanical δ-like potential in two spatial dimensions is mathematically ill-defined [4] and, eventually, the whole machinery of deficiency indices and subspaces should be involved in order to find all the self-adjoint extensions of the corresponding quantum Hamiltonian, what has not been done insofar[1]. A physically similar and mathematically consistent way to model point-like interactions is by means of vortex-like potentials [5] of the Aharonov-Bohm type [6].

[1] In [3] it is claimed that the model with a single δ-function impurity is ”essentially exactly solvable”. However, several approximations and formal manipulations, which are not mathematically clean, are involved. A rigorous treatment of the contact interaction in quantum mechanics [4] drives to different conclusions.

Anyway, apart from the detailed shape of the disorder potentials, the theoretical investigations concerning the calculation of the density of the quantum states, as well as of the electric conductivity, actually rely upon approximate methods and, notably, perturbative approaches [2],[3],[5],[7],[8]. Nonetheless, it turns out - up to our knowledge - that no exact non-perturbative solutions have been obtained, even in the simplest realizations of the above mentioned basic model of the two-dimensional electron ideal gas in the presence of disorder potentials.

It is the aim of the present paper to make a first step towards the filling of that lack, as we shall exhibit and discuss the exact solution for the quantum mechanical problem of a non-relativistic electron in the presence of a uniform electromagnetic field and of the Aharonov-Bohm vortex potential to describe one impurity. In spite of its first glance simplicity, the solution of the latter model is not trivial. As a matter of fact, while in the absence of the electric field the $O(2)$ rotational symmetry of the model naturally suggests the use of polar coordinates and of the symmetric gauge, the switching on of the symmetry breaking uniform electric field does indeed spoil that facility. Consequently, it appears to be extremely fruitful, in order to find the exact solution, to follow an algebraic method as well as to employ holomorphic coordinates [9]. Moreover, to reach our final goal it is necessary to perform some little mathematical *tour de force*, in order to become familiar with the realm of the self-adjoint extensions of the hermitean radial hamiltonian operators, *i.e.*, to specify the nature of their domains.

The present analysis shows that, in the absence of the electric field, a large - actually infinite - arbitrarity is allowed in the specification of the quantum radial hamiltonian operators. On the contrary, after the addition of a non-vanishing uniform electric field, the situation drastically changes: "localized eigenstates" are no longer allowed in the one-impurity model, all the eigenstates being improper and non-degenerate - just like in the impurity-free case, cause the Hamiltonian turns out to be essentially self-adjoint. This is the ultimate reason why the exact solution is unique. On the other hand, it is also found that the wave functions of the improper and non-degenerate conducting eigenstates are necessarily singular at the impurity position, a clear signal that the configuration manifold underlying the model is that of the one-punctured plane, *i.e.* topologically non-trivial.

All those above mentioned features of the model do represent the tools, thanks to which the total Hall conductance is computed to be the same as in the "classical" impurity-free problem, according again to the general consensus [3]. However, it should not be missed by the attentive reader that the detailed quantum mechanical microscopic mechanism, which eventually drives to the very same current and conductance in the zero- and one-impurity models, looks to be rather different from the ones usually acknowledged [2],[3],[5],[8]. In this sense we hope that the exact solution of the one-impurity model could shed some new perspective on the intimate microscopic nature leading on the onset of the Hall plateaus.

The paper is organized as follows. In section 2 we review the well known impurity-free model, albeit paying some special attention to the algebraic method that will be crucial in the analysis of the one-impurity case. Also the definitions of the current and conductance are shortly reconsidered. In section 3, the solution of the Landau's problem in the presence of the Aharonov-Bohm potential is reobtained [9] within the algebraic method, in the case in which the domain of the Hamiltonian is that of the regular wave functions. The generalizations to the case in which the domain is allowed to encompass singular although square integrable wave functions is the content of section 4, where the self-adjoint extensions of the radial Hamiltonians are discussed. The exact non-perturbative solution of the model in the presence of the uniform electric field is obtained in section 5.

Our conclusions are drawn in section 6, whereas some technical details are deferred to the appendix.

2. The exact solution without impurity.

In this section we review the exact solution for the non-relativistic quantum mechanical motion of a charged point-like particle of charge $-|e|$ and mass m (one electron) in *2+1* dimensions, in the presence of uniform - *i.e.* constant and homogeneous - electric and magnetic fields. Although this solution is very well known, we find it useful to reproduce the results within the so-called symmetric gauge and paying special attention to some algebraic methods that will be quite suitable later on, in order to treat the one-impurity problem. In so doing, we also establish our notations and conventions.

Our starting point is the two-dimensional euclidean massless Dirac's operator in the presence of a uniform magnetic field of strength $B > 0$ and orthogonal to the Ox_1x_2 plane. After choosing the symmetric gauge

$$A_j(x_1, x_2) = -\epsilon_{jl} x_l \frac{B}{2} \ , \quad j, l = 1, 2 \ , \quad \epsilon_{12} = 1 \ , \tag{2.1}$$

the Dirac's operator $i\mathcal{D} = i\sigma_j[\partial_j - i(|e|/c)A_j]$, $\sigma_{1,2}$ being the Pauli's matrices, can be recast into matrix form as

$$i\mathcal{D} = \begin{pmatrix} 0 & 2i\hbar\partial_z - \frac{i}{2}m\omega\bar{z} \\ 2i\hbar\partial_{\bar{z}} + \frac{i}{2}m\omega z & 0 \end{pmatrix} \ , \tag{2.2}$$

where $\omega \equiv (|e|B/mc)$ is the cyclotron's frequency and where we have set

$$z = \frac{x_1 + ix_2}{\lambda_B} = x + iy; \tag{2.3a}$$

$$x_1 = \lambda_B \frac{z + \bar{z}}{2} \ , \ x_2 = \lambda_B \frac{z - \bar{z}}{2i} \ ; \tag{2.3b}$$

$$\partial_z = (\lambda_B/2)\,(\partial_1 - i\partial_2) \ , \tag{2.3c}$$

$\lambda_B = \sqrt{\hbar c/|e|B}$ being the magnetic length. If we introduce the creation-destruction energy operators

$$\delta \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\right\} = \bar{\delta}^\dagger \ , \tag{2.4a}$$

$$\bar{\delta} \equiv i\sqrt{2}\left\{\partial_{z} - \frac{\bar{z}}{4}\right\} = \delta^\dagger \ , \tag{2.4b}$$

which fulfill the standard commutation relations $[\delta, \bar{\delta}] = 1$, then we can rewrite the Dirac's operator as

$$i\mathcal{D} = \sqrt{2}\frac{\hbar}{\lambda_B}\begin{pmatrix} 0 & \bar{\delta} \\ \delta & 0 \end{pmatrix} \ . \tag{2.5}$$

It is now immediate to realize that the Schrödinger-Pauli Hamiltonian for the spin-up component [2] in the presence of an additional uniform electric Hall field of a strength $E_H > 0$, along the positive Ox_1 direction, reads

$$H(E_H) = \frac{\hbar^2}{2m\lambda_B^2}\left(2\bar{\delta}\delta - \varrho\frac{z+\bar{z}}{2}\right) \ , \tag{2.6}$$

where the dimensionless parameter $\varrho \equiv 2(E_H/B)\sqrt{mc^2/\hbar\omega}$ has been introduced. The above hamiltonian operator, whose domain is that of the regular wave functions on the plane, turns out to be self-adjoint since, as we shall see, the eigenvalues are real and the eigenfunctions span a complete orthonormal set.

Now, there is a nice algebraic way to put the above Hamiltonian into diagonal form. To this aim, let us first introduce the following set of translated energy and degeneracy creation-destruction operators respectively: namely,

$$\delta_\varrho \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z-\varrho}{4}\right\} = \bar{\delta}_\varrho^\dagger \ , \tag{2.7a}$$

[2] Throughout this paper we shall always refer to the spin-up components of the wave functions, the generalization to the spin-down components being straightforward.

$$\bar{\delta}_\varrho \equiv i\sqrt{2}\left\{\partial_z - \frac{\bar{z}-\varrho}{4}\right\} = \delta_\varrho^\dagger \ , \tag{2.7b}$$

$$\theta_\varrho \equiv -i\sqrt{2}\left\{\partial_z + \frac{\bar{z}-\varrho}{4}\right\} = \bar{\theta}_\varrho^\dagger \ , \tag{2.7c}$$

$$\bar{\theta}_\varrho \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z-\varrho}{4}\right\} = \theta_\varrho^\dagger \ , \tag{2.7d}$$

which fulfil the operator algebra

$$\left[\delta_\varrho, \bar{\delta}_\varrho\right] = \left[\theta_\varrho, \bar{\theta}_\varrho\right] = 1 \ , \quad \left[\delta_\varrho, \theta_\varrho\right] = \left[\delta_\varrho, \bar{\theta}_\varrho\right] = 0 \ ; \tag{2.8}$$

then, it is a simple exercise to show that the hamiltonian operator (2.6) - up to the energy scale factor $(\hbar^2/2m\lambda_B^2)$ - can be cast into the form

$$\frac{2m\lambda_B^2}{\hbar^2} H(E_H) \equiv \mathtt{h}(\varrho) = 2\bar{\delta}_\varrho\delta_\varrho + i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2 \ . \tag{2.9}$$

The above expression for the Hamiltonian, together with the operator algebra (2.8), actually suggest that we can search for the eigenvectors of $\mathtt{h}(\varrho)$ as simultaneous eigenstates of the "Landau-like" Hamiltonian $2\bar{\delta}_\varrho\delta_\varrho$ and of the operator

$$\mathtt{T}(\varrho) \equiv i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2 = -\varrho\left(\frac{\lambda_B p_2}{\hbar} + \frac{x_1 - \varrho\lambda_B}{2\lambda_B} + \frac{3}{4}\varrho\right) \ , \tag{2.10}$$

which admits a continuous spectrum and represents the combined effect of a translation along the Ox_2-axis and a gauge transformation. As a matter of fact, if we introduce the real number $p_\perp \equiv (\lambda_B p_2/\hbar)$ where p_2 is the transverse momentum - orthogonal to the electric field - we obtain

$$\mathtt{T}(\varrho)\exp\left\{iy\left(p_\perp - \frac{x}{2}\right)\right\} = \left(-\varrho p_\perp - \frac{1}{4}\varrho^2\right)\exp\left\{iy\left(p_\perp - \frac{x}{2}\right)\right\} \ . \tag{2.11}$$

It follows therefrom, that the Hamiltonian (2.9) has a continuous non-degenerate spectrum whose eigenvalues are given by

$$\varepsilon_{n,p_\perp} = 2n - \varrho p_\perp - \frac{1}{4}\varrho^2 \ , \qquad n+1 \in \mathbf{N} \ , \quad p_\perp \in \mathbf{R} \ , \tag{2.12}$$

which reproduce the well known electric field splitting of the Landau's bands.

In order to exhibit the eigenfunctions, it tuns out to be convenient, owing to our later purposes, to consider the Bargmann-Segal basis in the holomorphic representation, which is constructed out of the cyclic ground state-vector $|0,0;\varrho\rangle$ defined by

$$\delta_\varrho\,|0,0;\varrho\rangle = \theta_\varrho\,|0,0;\varrho\rangle = 0\ , \quad \langle 0,0;\varrho|0,0;\varrho\rangle = 1\ , \tag{2.13}$$

whose holomorphic representation reads

$$\langle z\bar{z}|0,0;\varrho\rangle \equiv \varphi_{0,0}(z,\bar{z};\varrho) = \sqrt{\frac{1}{2\pi}}\exp\left\{-\frac{1}{4}(\bar{z}-\varrho)(z-\varrho)\right\}\ . \tag{2.14}$$

The generic element of the basis can be thereof written as

$$|n,k\rangle = \sqrt{\frac{1}{n!k!}}\bar{\delta}_\varrho^n\bar{\theta}_\varrho^k\,|0,0;\varrho\rangle\ , \qquad n+1, k+1 \in \mathbf{N}\ , \tag{2.15}$$

its holomorphic representation being

$$\begin{gathered}\langle z\bar{z}|n,k;\varrho\rangle \equiv \varphi_{n,k}(z,\bar{z};\varrho) = \sqrt{\frac{1}{2\pi}}\exp\left\{-\frac{1}{4}(\bar{z}-\varrho)(z-\varrho)\right\}\times \\ \left(-i\frac{\bar{z}-\varrho}{\sqrt{2}}\right)^n\left(i\frac{z-\varrho}{\sqrt{2}}\right)^k\sum_{h=0}^{\infty}\left(-\frac{2}{|z-\varrho|^2}\right)^h\frac{\sqrt{n!k!}}{h!\Gamma(n+1-h)\Gamma(k+1-h)}\ .\end{gathered} \tag{2.16}$$

As it is well known, the above states span a complete orthonormal set. Our aim is now to express the eigenfunctions of the Hamiltonian (2.9) as linear combinations of the above Bargmann-Segal state vectors. Let us search therefore the eigenfunctions of (2.9) in the form

$$\psi_{n,p_\perp}(x,y;\varrho) = \sum_{k=0}^{\infty} c_k^{(n)}(p_\perp)\varphi_{n,k}(z,\bar{z};\varrho)\ . \tag{2.17}$$

It is not difficult to check that, in order to fulfil the eigenvalue problem

$$\mathtt{h}(\varrho)\psi_{n,p_\perp}(x,y;\varrho) = \left(2n - \varrho p_\perp - \frac{1}{4}\varrho^2\right)\psi_{n,p_\perp}(x,y;\varrho)\ , \tag{2.18}$$

the coefficients $c_k^{(n)}(p_\perp)$ must satisfy the following recursive relations: namely,

$$\begin{aligned} - i\sqrt{2}\tilde{p}c_0^{(n)} &= c_1^{(n)} \ ; \quad \tilde{p} \equiv p_\perp - \frac{1}{2}\varrho \ , \\ - i\sqrt{2}\tilde{p}c_k^{(n)} &= \sqrt{k+1}c_{k+1}^{(n)} - \sqrt{k}c_{k-1}^{(n)} \ , \qquad (k \in \mathbf{N}) \end{aligned} \tag{2.19}$$

whose solution is well known to be

$$c_k^{(n)}(\tilde{p}) = (-i)^k u_k(\tilde{p}) \ , \qquad k+1, n+1 \in \mathbf{N} \ , \tag{2.20}$$

where $\{u_k(\tilde{p}),\ k+1 \in \mathbf{N}\}$ is the complete orthonormal set of the Hermite's functions [10]. From orthonormality and completeness of the sets (2.16) and (2.20), it readily follows that the improper eigenfunctions (2.17) are complete and orthonormalized in the continuum according to

$$\langle \psi_{n,p_\perp}(\varrho) | \psi_{m,q_\perp}(\varrho) \rangle = \delta_{n,m}\delta(p_\perp - q_\perp) \ , \qquad p_\perp, q_\perp \in \mathbf{R} \ , n+1, m+1 \in \mathbf{N} \ . \tag{2.21}$$

As a further remark, we want to briefly comment about the transition to the Landau (or asymmetric) gauge eigenfunctions. The latter ones are very well known to be given by

$$\chi_{n,p_\perp}(x,y;\varrho) = \sqrt{\frac{1}{2\pi}} u_n\left(x - \frac{1}{2}\varrho - p_\perp\right)\exp\{iyp_\perp\} \ , \tag{2.22}$$

from which it is possible to directly verify that the eigenfunctions within the two gauge choices are merely related by the gauge phase factor $\exp\{-(i/2)yx\}$, as it does, since the spectrum is non-degenerate. It means that

$$\begin{aligned} \langle xy | \psi_{n,p_\perp}(\varrho) \rangle \equiv \psi_{n,p_\perp}(x,y;\varrho) &= \chi_{n,p_\perp}(x,y;\varrho)\exp\left\{-i\frac{xy}{2}\right\} \\ &= u_n\left(x - \frac{1}{2}\varrho - p_\perp\right)\exp\left\{iyp_\perp - i\frac{xy}{2}\right\}\sqrt{\frac{1}{2\pi}} \ . \end{aligned} \tag{2.23}$$

The reason why we have developed the much less direct procedure previously described is because the latter one, although academic at this stage, will be crucial in order to treat and explicitely solve the corresponding one-impurity problem.

Finally, we briefly reconsider the derivation of the electron current and conductance within the "classical" impurity-free situation. According to elementary quantum mechanics, the planar current operator is given by

$$\hat{J}_{1,2} = \frac{|e|\hbar}{m\lambda_B}\hat{P}_{1,2} \ , \tag{2.24}$$

where

$$\hat{P}_1 \equiv \left(-i\frac{\partial}{\partial x} + \frac{y}{2}\right) \ , \quad \hat{P}_2 \equiv \left(-i\frac{\partial}{\partial y} - \frac{x}{2}\right) \ . \tag{2.25}$$

Now, if we consider an electron in the n-th Landau's band, it is in general described by a proper - normalizable - wave packet of the kind

$$\psi_{n,f}(x,y;\varrho) = \int_{-\infty}^{+\infty} dp_\perp \ f(p_\perp)\psi_{n,p_\perp}(x,y;\varrho) \ , \tag{2.26}$$

with f arbitrary function normalized to one, *i.e.*, $\int_{-\infty}^{+\infty} dp_\perp \ |f(p_\perp)|^2 = 1$.

It follows that the average current carried by such a state is given by

$$J_{1,2}^{(n)}[f] = \left\langle \psi_{n,f} \left| \hat{J}_{1,2} \right| \psi_{n,f} \right\rangle \ . \tag{2.27}$$

Since we have

$$\hat{P}_1 = -\frac{1}{\sqrt{2}}\left(\delta_\varrho + \bar{\delta}_\varrho\right) \ , \tag{2.28a}$$

$$\hat{P}_2 = \frac{i}{\sqrt{2}}\left(\delta_\varrho - \bar{\delta}_\varrho + \frac{i}{\sqrt{2}}\varrho\right) \ , \tag{2.28b}$$

it is immediate to verify from eq.s (2.17) or (2.23) that we obtain

$$J_1^{(n)}[f] = 0 \ , \qquad J_2^{(n)}[f] = -\frac{|e|c}{B}E_H \ , \tag{2.29}$$

which turn out to be independent from the form of the wave packet. This amounts to say that each improper eigenstate of the Hamiltonian (2.6) carries the same current

$(0, -|e|c(E_H/B))$. It follows that the "classical" Hall's conductance of each eigenstate is provided by

$$\sigma_{xy} = -\frac{e^2}{h}\Gamma_L^{-1} \ , \tag{2.30}$$

where the usual Landau's levels degeneracy factor is

$$\Gamma_L \equiv \frac{1}{2\pi\lambda_B^2} = \frac{|e|B}{hc} \ . \tag{2.31}$$

According to the above described simple property, it turns out that the total Hall conductance of an ideal electron gas in a pure sample is proportional to the filling factor $\nu = (\mathtt{n}/\Gamma_L)$, where $\mathtt{n}$ denotes the number of electrons *per* unit area.

3. The Landau's problem in the presence of the AB-vortex.

The problem of a point-like charged particle on the plane in the presence of a uniform magnetic field and one Aharonov-Bohm point-like singularity - the AB-vortex - has been already studied in the Literature [9]. By the way, it turns out that the AB-vortex faithfully describes [5] the presence of some localized impurity within the Hall's sample. It is quite instructive to resolve this problem by means of a suitable algebraic method. In so doing, in fact, it is possible to unravel some interesting features of the exact solutions, which have been not yet discussed insofar, up to our knowledge, but will be crucial in order to provide the exact solution in the presence of an additional uniform electric field.

The gauge potential, in the symmetric gauge, is now given by

$$A_j(x_1, x_2) = -\epsilon_{jl}x_l \left(\frac{B}{2} - \frac{(\phi/2\pi)}{x_1^2 + x_2^2} \right) \ , \tag{3.1}$$

in which the flux-parameter $\phi > 0 (< 0)$ means that the vortex magnetic field, located at the origin, is anti-parallel (parallel) to the uniform magnetic field $B > 0$. After introduction of the quantum flux unity $\phi_0 \equiv (hc/|e|)$ and of the dimensionless parameter $\alpha \equiv (\phi/\phi_0)$, it can be easily shown that the rescaled Schrödinger-Pauli Hamiltonian for the upper spinor component takes the form

$$\mathtt{h}(\alpha) = 2\bar{\delta}(\alpha)\delta(\alpha) \ , \tag{3.2}$$

where the singular creation-destruction energy operators appear to be

$$\delta(\alpha) \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\delta}^{\dagger}(\alpha) \ , \tag{3.3a}$$

$$\bar{\delta}(\alpha) \equiv i\sqrt{2}\left\{\partial_{z} - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right)\right\} = \delta^{\dagger}(\alpha) \ , \tag{3.3b}$$

with $\gamma \equiv (\bar{z}z/2)$. The singularity at $\gamma = 0$ in the foregoing expressions is understood in the sense of the tempered distributions [12]: namely,

$$\begin{aligned} \frac{1}{[\gamma]} &\equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\ln\sqrt{x^2+y^2}\right)^2 + C\delta(x)\delta(y) \\ &= \frac{1}{4}\triangle \ln^2(\bar{z}z) + C\delta^{(2)}(\bar{z}, z) \ , \end{aligned} \tag{3.4}$$

where C is an arbitrary constant, whose presence ensures *naïve* scaling behavior of the tempered distribution itself, *i.e.*, $1/[c\gamma] = (1/c)(1/[\gamma]), \ c > 0$. To start with, let us consider the domain of the operators (3.3) to be the Besov's space $\mathcal{T}(\mathbf{R}^2) = \{f \in \mathcal{S}(\mathbf{R}^2) |\ f(0) = 0\}$, which is dense in $L^2(\mathbf{R}^2)$.

To be definite and without loss of generality, we shall choose in the sequel $-1 < \alpha < 0$, corresponding to parallel uniform and vortex magnetic fields. As a matter of fact, it is well known that only the non-integer part of α is relevant, its integer part being always reabsorbed by means of a single-valued gauge transformation.

In order to fully solve the eigenvalue problem, it is convenient to introduce also the associated singular creation-destruction degeneracy operators

$$\theta(\alpha) \equiv -i\sqrt{2}\left\{\partial_{z} + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \bar{\theta}^{\dagger}(\alpha) \ , \tag{3.5a}$$

$$\bar{\theta}(\alpha) \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right)\right\} = \theta^{\dagger}(\alpha) \ , \tag{3.5b}$$

always acting on the same domain $\mathcal{T}(\mathbf{R}^2)$, in such a way that the following simple operator algebra still holds true for any $-1 < \alpha \leq 0$: namely,

$$[\delta(\alpha), \bar{\delta}(\alpha)] = [\theta(\alpha), \bar{\theta}(\alpha)] = 1 \ , \quad [\delta(\alpha), \theta(\alpha)] = [\delta(\alpha), \bar{\theta}(\alpha)] = 0 \ . \tag{3.6}$$

We notice that, in order to reproduce the foregoing algebra, it is essential to employ the definition (3.4). As a matter of fact, eq. (3.4) guarantees the *naïve* action of the dilatation operator

$$\mathtt{D}\frac{1}{[\gamma]} = -2\frac{1}{[\gamma]} \ , \quad \mathtt{D} \equiv z\partial_z + \bar{z}\partial_{\bar{z}} \ , \tag{3.7}$$

whence it is an easy exercise to check the algebra (3.6).

Now, owing to $O(2)$-symmetry of the problem, we can search for common eigenfunctions of the rescaled Hamiltonian (3.2) and of the angular momentum operator

$$\mathtt{L} \equiv \hbar(z\partial_z - \bar{z}\partial_{\bar{z}}) = \hbar\{|\alpha|\mathbf{1} + \bar{\theta}(\alpha)\theta(\alpha) - \bar{\delta}(\alpha)\delta(\alpha)\} \ , \tag{3.8}$$

which manifestly commutes with $\mathtt{h}(\alpha)$.

Let us consider in the present section the case in which the domain of the rescaled Hamiltonian (3.2) is $\mathcal{S}(\mathbf{R}^2)$ - wave functions regular at the origin - which is dense in $L^2(\mathbf{R}^2)$ and provides the standard solution given in the literature [9]. The eigenstates are naturally separated into two classes which we call integer-valued energy eigenstates (I.V.E.), which form an infinite degenerate set, and real-valued energy eigenstates (R.V.E.) whose degeneracy is always finite.

(I.V.E.) This kind of eigenstates correspond to integer eigenvalues $\breve{\varepsilon}_n = 2n \ , \ n+1 \in \mathbf{N}$, of the rescaled Hamiltonian. They just correspond to the Landau's bands in the absence of the impurity, up to the fact that their populations is lowered owing to the presence of the AB-vortex impurity. They are constructed as follows.

Let us look for the solution of the ground state equation

$$\delta(\alpha) f_{0,1} = 0 \; ; \tag{3.9}$$

its most general form is provided by

$$f_{0,1}(z,\bar{z}) = f_1(z)\gamma^{-|\alpha|/2}\exp\{-\gamma/2\} \; , \tag{3.10}$$

in which $f_1(z)$ is $\mathcal{O}(z)$ for small z , in order that $f_{0,1}$ belongs to the domain $\mathcal{T}(\mathbf{R}^2)$ of the destruction operator. It is now easy to realize that the lowest Landau's band is spanned by the orthonormal eigenfunctions

$$\langle z\bar{z}|0<k;\check{\alpha}\rangle = \frac{(iz/\sqrt{2})^k}{\sqrt{2\pi\Gamma(k+1+\alpha)}}\gamma^{-|\alpha|/2}\exp\{-\gamma/2\} \; , \qquad k\in\mathbf{N} \; , \tag{3.11}$$

corresponding to

$$|0<k;\check{\alpha}\rangle = \frac{[\bar{\theta}(\alpha)]^{k-1}}{\sqrt{(\alpha+2)_{k-1}}}\,|0<1;\check{\alpha}\rangle \; , \qquad k\in\mathbf{N} \; , \tag{3.12}$$

where $(\alpha+2)_{k-1} \equiv (\alpha+2)(\alpha+3)\ldots(\alpha+k) = [\Gamma(\alpha+1+k)/\Gamma(\alpha+2)]$ denotes the usual Pochhammer symbol.

Therefore, it is immediate to gather that the n-th Landau's band of rescaled energy $\check{\varepsilon}_n = 2n$ is spanned by the eigenstates

$$|n<k;\check{\alpha}\rangle = \sqrt{\frac{\Gamma(\alpha+2)}{(n!)\Gamma(k+1+\alpha)}}[\bar{\delta}(\alpha)]^n[\bar{\theta}(\alpha)]^{k-1}\,|0<1;\check{\alpha}\rangle \; , \quad k\geq n+1\in\mathbf{N} \; , \tag{3.13}$$

which exhibit the infinite degeneracy of the Landau's bands, the degeneracy being labelled by the quantum number $k\geq n+1$. Notice that the integer-valued energy bands contain an infinite number of states, although $n+1$ states less than the corresponding ordinary Landau's band in the absence of the AB-vortex impurity. We remark that all the I.V.E.

actually belong to $\mathcal{T}(\mathbf{R}^2)$, since it is easy to check that their holomorphic representations

$$\langle z\bar{z}|n<k;\check{\alpha}\rangle = (-1)^n\sqrt{\frac{n!}{2\pi\Gamma(k+\alpha+1)}}\left(i\frac{z}{\sqrt{2}}\right)^{k-n}\gamma^{-|\alpha|/2}\exp\{-\gamma/2\}L_n^{(k-n+\alpha)}(\gamma)$$
$$\equiv \Psi_{n<k}(z,\bar{z})\ , \qquad k\geq n+1\in\mathbf{N}\ , \tag{3.14}$$

do vanish at the origin, $L_n^{(\beta)}$ being the generalized Laguerre's polynomials. We remark that, as it appears to be manifest from the above expression, the items $k=0,1,2,\ldots,n$ are forbidden as they drive outside either the domain of the Hamiltonian ($k=n$) or even outside $L^2(\mathbf{R}^2)$ ($k=0,1,\ldots,n-1$).

(R.V.E.) These eigenstates correspond to non-integer eigenvalues $\hat{\varepsilon}_n = 2(n+|\alpha|)$ of the rescaled Hamiltonian. Their construction can be easily obtained along the very same line as in the afore mentioned procedure.

The starting point is the basic wave function

$$\langle z\bar{z}|0,0;\hat{\alpha}\rangle = \frac{\gamma^{|\alpha|/2}\exp\{-\gamma/2\}}{\sqrt{2\pi\Gamma(1-\alpha)}}\ , \tag{3.15}$$

which enjoys

$$\theta(\alpha)\,|0,0;\hat{\alpha}\rangle = 0\ , \tag{3.16a}$$

$$\mathtt{h}(\alpha)\,|0,0;\hat{\alpha}\rangle = 2|\alpha|\,|0,0;\hat{\alpha}\rangle\ , \tag{3.16b}$$

and belongs to $\mathcal{T}(\mathbf{R}^2)$. Now, it is evident that the n-th excited R.V.E. are $(n+1)$-times degenerate and given by

$$|n\geq k;\hat{\alpha}\rangle = \sqrt{\frac{\Gamma(1-\alpha)}{(k!)\Gamma(n+1-\alpha)}}[\bar{\delta}(\alpha)]^n[\bar{\theta}(\alpha)]^k\,|0,0;\hat{\alpha}\rangle\ , \quad 0\leq k\leq n\ , \tag{3.17}$$

the corresponding wave functions being

$$\langle z\bar{z}|n\geq k;\hat{\alpha}\rangle = (-1)^k\sqrt{\frac{k!}{2\pi\Gamma(n-\alpha+1)}}\left(-i\frac{\bar{z}}{\sqrt{2}}\right)^{n-k}\gamma^{|\alpha|/2}\exp\{-\gamma/2\}L_k^{(n-k-\alpha)}(\gamma)$$
$$\equiv \Psi_{n\geq k}(z,\bar{z})\ , \quad n+1\in\mathbf{N}\ , \quad 0\leq k\leq n\ , \tag{3.18}$$

which, again, belong to the above specified domain of $\mathtt{h}(\alpha)$ iff the degeneracy quantum number k does not exceed the energy quantum number n.

Some few comments are now in order. First, according to eq. (3.8), the angular momentum of the generic eigenstate is given by $\ell = \hbar(k-n)$. This means that I.V.E. correspond to positive values of the angular momentum, whereas R.V.E. correspond to negative or vanishing eigenvalues. Second, it is easily understood that the above set of eigenstates is a complete orthonormal basis, as they are into a one-to-one unitary correspondence with the states $\varphi_{n,k}(z,\bar{z};\varrho=0)$ of eq. (2.16). Third, the I.V.E. and R.V.E. wave functions are connected by the duality relation

$$\Psi_{n<k}(z,\bar{z}|\alpha) = \Psi^*_{k<n}(z,\bar{z}|-\alpha) \ . \tag{3.19}$$

4. Self-adjoint extensions of the Hamiltonian.

We have considered insofar the rescaled Hamiltonian $\mathtt{h}(\alpha)$ to be defined on the domain of the regular square-integrable wave functions. However, as it is well known, this is not the most general case. Let us now consider, therefore, different quantum Hamiltonians, corresponding to different self-adjoint hamiltonian operators, whose differential operator is always given by eq. (3.2), but the domain is now allowed to contain wave functions with square integrable singularities at the origin - *i.e.* at the AB-vortex position. This procedure is the mathematically correct way to introduce in the context contact-interaction - or point-like interaction. In physical terms, it is equivalent to *naïvely* add some kind of δ-like potential [3] to the classical Hamiltonian. We have to stress in fact that, strictly speaking, δ-like potential are ill-defined in two and three spatial dimensions [4],[13] and the proper way to encompass the possibility of contact-interaction is by means of the analysis of the self-adjoint extensions of the quantum Hamiltonian. In particular, the solution we have discussed in the previous section, *i.e.* the case of the Hamiltonian whose domain is that of the regular wave functions, can be thought of as the pure AB interaction in the absence of contact-interaction. The presence of a particular square integrable singularity of the wave funtion at the vortex position will select some new quantum Hamiltonian, which will describe the presence of a specific contact-interaction. What we shall see in the sequel is that there is an infinite number of such Hamiltonians, which are perfectly legitimate and turn out to describe different physics, as they are characterized by different spectra and degeneracies.

Let us begin by describing the simplest case and consider the state $|0,0;\breve{\alpha}\rangle$ which is defined by

$$\begin{aligned} \delta(\alpha)\,|0,0;\breve{\alpha}\rangle = 0\ , \qquad z\neq 0\ , \\ \mathtt{L}\,|0,0;\breve{\alpha}\rangle = 0\ , \qquad z\neq 0\ , \end{aligned} \tag{4.1}$$

whose wave function in the holomorphic representation reads

$$\langle z\bar{z}|0,0;\breve{\alpha}\rangle = \frac{\gamma^{-|\alpha|/2}\exp\{-\gamma/2\}}{\sqrt{2\pi\Gamma(1+\alpha)}}\ . \tag{4.2}$$

The wave function of this state is singular at the origin, although square integrable on the whole plane and, owing to this, it does not belong to the domain of the Hamiltonian $\mathtt{h}(\alpha)$ previously discussed. Starting from the singular ground state (4.1), it is easy to construct the whole infinite set of excited singular energy eigenstates $|n,n;\breve{\alpha}\rangle$ whose corresponding wave functions are

$$\begin{aligned} \langle z\bar{z}|n,n;\breve{\alpha}\rangle &= (-1)^n\sqrt{\frac{n!}{2\pi\Gamma(n+\alpha+1)}}\gamma^{-|\alpha|/2}\exp\{-\gamma/2\}L_n^{(\alpha)}(\gamma) \\ &\equiv \Phi_{n,n}(z,\bar{z})\ , \quad n+1\in\mathbf{N}\ , \end{aligned} \tag{4.3}$$

which are singular at the origin and square integrable on the plane. We notice that the wave functions (4.3) are real and connected by the duality relation (3.19) with the regular wave functions $\Psi_{n,n}(z,\bar{z})$.

Now, since the singular wave functions $\Phi_{n,n}(z,\bar{z})$ turn out to be eigenfunctions, when $z \neq 0$, of the differential operator $\mathtt{h}(\alpha)$ as well as of the angular momentum $\mathtt{L}$, with eigenvalues $\check{\varepsilon}_n = 2n$ and $\ell = 0$ respectively, we can see that a new self-adjoint Hamiltonian $\mathtt{H}^{(0)}(\alpha)$ arises. It is at the best characterized by its spectral decomposition

$$\mathtt{H}^{(0)}(\alpha) = \sum_{n=1}^{\infty} \left\{ 2n \sum_{k=n}^{\infty} \check{P}_{n\le k}(\alpha) + 2(n+|\alpha|) \sum_{k=0}^{n-1} \hat{P}_{n>k}(\alpha) \right\} , \tag{4.4}$$

together with its kernel, *i.e.*, the zero-modes which span the lowest Landau's band,

$$\mathrm{Ker}[\mathtt{H}^{(0)}(\alpha)] = \{ |0 \le k; \check{\alpha}\rangle \ ; k+1 \in \mathbf{N} \} , \tag{4.5}$$

where the I.V.E. and R.V.E. projectors have been introduced respectively: namely,

$$\check{P}_{n\le k}(\alpha) \equiv |n \le k; \check{\alpha}\rangle \langle n \le k; \check{\alpha}| \ , \quad \hat{P}_{n>k}(\alpha) \equiv |n > k; \hat{\alpha}\rangle \langle n > k; \hat{\alpha}| \ . \tag{4.6}$$

Notice that the new Hamiltonian is still rotational invariant since

$$[\mathtt{H}^{(0)}(\alpha), \mathtt{L}] = 0 \ . \tag{4.7}$$

We remark that the eigenvalue $\hat{\varepsilon}_0 = 2|\alpha|$ does not belong anymore to the spectrum of $\mathtt{H}^{(0)}(\alpha)$, at variance with the previous Hamiltonian $\mathtt{h}(\alpha)$. Furthermore, the degeneracy of each one of the remaining non-integer energy levels is lowered by one unity and, correspondingly, the whole infinite set of the energy eigenstates with vanishing angular momentum is removed from the real-valued and included into the integer-valued sector. Consequently, the degeneracy of each Landau's band of the Hamiltonian $\mathtt{H}^{(0)}(\alpha)$ is increased by one unity. Finally, it is immediate to gather that the eigenstates of the Hamiltonian $\mathtt{H}^{(0)}(\alpha)$ actually span a complete orthonormal set. To show this, let us first write the singular states $|n,n;\check{\alpha}\rangle$ as L^2-expansions over the regular basis $\{|j,j;\hat{\alpha}\rangle \ ; j+1 \in \mathbf{N}\}$ in the appropriate subspace of vanishing angular momentum. ¿From eq.s (3.14) and (4.3) we obtain

$$\Phi_{n,n}(z,\bar{z}) = \sum_{j=0}^{\infty} \check{c}_{n,0}^{\,j} \Psi_{j,j}(z,\bar{z}) \ , \tag{4.8}$$

with

$$\check{c}_{n,0}^{\,j} \equiv \langle j,j;\hat{\alpha}|n,n;\check{\alpha}\rangle = \binom{j-\alpha}{n}\binom{n+\alpha}{j}\sqrt{\frac{j!n!}{\Gamma(j-\alpha+1)\Gamma(n+\alpha+1)}} \ . \tag{4.9}$$

Now, since we have by construction - see eq.s (3.18) and (4.3) -

$$\langle n,n;\hat{\alpha}|m,m;\hat{\alpha}\rangle = \delta_{n,m} = \langle n,n;\check{\alpha}|m,m;\check{\alpha}\rangle \ , \quad n+1\in\mathbf{N},\ m+1\in\mathbf{N} \ , \tag{4.10}$$

it follows that the coefficients $\check{c}_{n,0}^{\,j}$ are the entries of a unitary change of basis in the infinite dimensional subspace of zero angular momentum. Consequently, according to the fundamental theorem, the Hamiltonian $\mathtt{H}^{(0)}(\alpha)$ is self-adjoint since it admits a complete orthonormal set of eigenvectors.

It is not difficult now to realize how to go one more step further. Let us in fact consider the generic subspace of negative angular momentum $\ell = \hbar(k-n) < 0$, with $0 \le k < n \in \mathbf{N}$. If the domain of the Hamiltonian is that of the regular wave functions, the latter subspace is spanned by the real-valued energy eigenstates $|n>k;\hat{\alpha}\rangle$ of increasing non-integer energies $\hat{\varepsilon}_n = 2(n+|\alpha|)$, $n\in\mathbf{N}$. However, if we relax regularity at the origin of the wave functions, we can set up a different basis in the subspace of fixed negative angular momentum $\ell = -\hbar l,\ l = 1,\dots,n\in\mathbf{N}$: namely,

$$|n>n-l;\check{\alpha}\rangle \equiv \sum_{j=0}^{\infty}\check{c}_{n,l}^{\,j}\,|j>j-l;\hat{\alpha}\rangle \ , \qquad l=1,\dots,n\in\mathbf{N} \ . \tag{4.11}$$

where

$$\begin{aligned}\check{c}_{n,l}^{\,j} &\equiv \langle j>j-l;\hat{\alpha}|n>n-l;\check{\alpha}\rangle \\ &= \binom{j-\alpha}{n}\binom{n+\alpha-l}{j-l}\sqrt{\frac{(j-l)!n!}{\Gamma(j-\alpha+1)\Gamma(n-l+\alpha+1)}} \ .\end{aligned} \tag{4.12}$$

The above eq. (4.11) uniquely defines a state vector $\forall n+1\in\mathbf{N}$, owing to the Riesz-Fisher theorem, since it can be actually verified - see appendix - that

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\check{c}_{n,l}^{\,k}\right|^2 &= \frac{\{(n+\alpha)(\alpha)_n\}^2}{n!\Gamma(1-\alpha)\Gamma(n-l+\alpha+1)[(-n-\alpha)_l]^2}\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(k-l)!(n+\alpha-k)^2} \\ &= 1 \ .\end{aligned} \tag{4.13}$$

The corresponding wave functions in the holomorphic representation

$$\Phi_{n>n-l}(z,\bar{z}) = \sum_{j=0}^{\infty} \check{c}_{n,l}^{\,j} \hat{\psi}_{j,j-l}(z,\bar{z};\alpha) \ , \qquad l = 1,\ldots,n \ , \tag{4.14}$$

belong to $L^2(\mathbf{R}^2)$ by construction, are singular at the AB-vortex position and are eigenfunctions of the differential operator (3.2) with integer eigenvalues $\check{\varepsilon}_n = 2n$ and of the angular momentum operator (3.8) with a negative eigenvalue $\ell = -\hbar l$. The net result of this construction is that, by relaxing the condition of the regularity of the wave functions at the AB-vortex position - which specifies a particular domain of the quantum Hamiltonian - it is possible to set up different quantum Hamiltonians, with different spectra and degeneracies, after shifting an infinite set of states (actually orthonormal and complete in the subspaces of fixed angular momenta) from the R.V.E. sector to the I.V.E. one. We can play this game *ad nauseam* and state the following

Lemma $(-)$:

in any subspace of fixed negative angular momentum $\ell = -\hbar l$, $l \in \mathbf{N}$ there are two options in order to specify the quantum radial Hamiltonian: if the domain is that of regular wave functions we have

$$\mathtt{h}_l(\alpha) = \sum_{n=l}^{\infty} 2(n+|\alpha|)\hat{P}_{n>n-l}(\alpha) \ ; \tag{4.15}$$

alternatively, if the domain is that of the wave functions which are square integrable on the plane, although singular at the impurity's position, we have

$$\mathtt{H}_l(\alpha) = \sum_{n=l}^{\infty} 2n\check{P}_{n>n-l}(\alpha) \ , \tag{4.16}$$

where

$$\check{P}_{n>n-l}(\alpha) \equiv |n>n-l;\check{\alpha}\rangle \, \langle n>n-l;\check{\alpha}| \ , \tag{4.17}$$

the state $|n>n-l;\check{\alpha}\rangle$ being given by eq. (4.11).

A quite similar construction can be done for the quantum radial Hamiltonians corresponding to positive angular momenta $\ell = \hbar l,\ l \in \mathbf{N}$. As a matter of fact, we can start from the subspace $l = 1$ in which a basis $\{|n < n+1; \hat{\alpha}\rangle\ ,\ n+1 \in \mathbf{N}\}$, other than the regular one, *i.e.*, $\{|n < n+1; \check{\alpha}\rangle\ ,\ n+1 \in \mathbf{N}\}$, can be easily obtained out of wave functions square integrable on the plane although singular at the origin: namely,

$$\begin{aligned}\langle z\bar{z}|n < n+1; \hat{\alpha}\rangle &= (-1)^{n+1}\sqrt{\frac{(n+1)!}{2\pi\Gamma(n+1-\alpha)}}\left(i\frac{z}{\sqrt{2}}\right)\gamma^{-1+|\alpha|/2}\exp\{-\gamma/2\}L_{n+1}^{(-1-\alpha)}(\gamma)\\ &\equiv \Phi_{n<n+1}(z,\bar{z})\ , \quad n+1 \in \mathbf{N}\ .\end{aligned} \tag{4.18}$$

It can be easily verified that the above singular wave functions are eigenfunctions, when $z \neq 0$, of the hamiltonian differential operator (3.2) with non-integer eigenvalues $\hat{\varepsilon}_n = 2(n + |\alpha|)$. Furthermore, we can expand them on the regular basis as

$$\Phi_{n<n+1}(z,\bar{z}) = \sum_{j=0}^{\infty} \hat{c}_{n,1}^{\,j}\,\Psi_{j<j+1}(z,\bar{z})\ , \tag{4.19}$$

with

$$\begin{aligned}\hat{c}_{n,1}^{\,j} &\equiv \langle j < j+1; \check{\alpha}|n < n+1; \hat{\alpha}\rangle\\ &= \binom{j+\alpha+1}{n+1}\binom{n-\alpha}{j}\sqrt{\frac{j!(n+1)!}{\Gamma(j+\alpha+2)\Gamma(n-\alpha+1)}}\ .\end{aligned} \tag{4.20}$$

It follows therefrom that, as before, we can construct the singular basis in the generic partial subspace of positive angular momentum $\ell = \hbar l,\ l \in \mathbf{N}$ according to

$$|n < n+l; \hat{\alpha}\rangle \equiv \sum_{j=0}^{\infty} \hat{c}_{n,l}^{\,j}\,|j < j+l; \check{\alpha}\rangle\ , \qquad l \in \mathbf{N}\ , \tag{4.21}$$

where

$$\begin{aligned}\hat{c}_{n,l}^{\,j} &\equiv \langle j < j+l; \check{\alpha}|n < n+l; \hat{\alpha}\rangle\\ &= \binom{j+\alpha+l}{n+l}\binom{n-\alpha}{j}\sqrt{\frac{j!(n+l)!}{\Gamma(j+l+\alpha+1)\Gamma(n-\alpha+1)}}\ .\end{aligned} \tag{4.22}$$

Again, it can be readily verified that

$$\sum_{j=0}^{\infty}\left|\hat{c}_{n,l}^{\,j}\right|^2 = \frac{(n-\alpha)^2\left\{(-\alpha)_n(1+\alpha)_l\right\}^2}{(n+l)!\Gamma(n-\alpha+1)\Gamma(\alpha+l+1)}\sum_{k=0}^{\infty}\frac{(\alpha+l+1)_k}{k!(n-\alpha-k)^2} = 1\ , \tag{4.23}$$

which means that, according to the Riesz-Fisher theorem, the expansion (4.21) uniquely defines a state vector in the Hilbert space, whose corresponding singular wave function is

$$\Phi_{n<n+l}(z,\bar{z}) = \sum_{j=0}^{\infty} \hat{c}_{n,l}^{\,j} \check{\psi}_{j,j+l}(z,\bar{z};\alpha) \ , \qquad l \in \mathbf{N} \ . \tag{4.24}$$

It turns out therefrom that we can state the following

Lemma (+):

in any subspace of fixed positive angular momentum $\ell = \hbar l$, $l \in \mathbf{N}$ there are two options in order to specify the quantum radial Hamiltonian: if the domain is that of regular wave functions we have

$$\mathtt{h}_l(\alpha) = \sum_{n=l}^{\infty} 2n \check{P}_{n<n+l}(\alpha) \ ; \tag{4.25}$$

alternatively, if the domain is that of the wave functions which are square integrable on the plane, although singular at the impurity's position, we have

$$\mathtt{H}_l(\alpha) = \sum_{n=l}^{\infty} 2(n+|\alpha|) \hat{P}_{n<n+l}(\alpha) \ , \tag{4.26}$$

where

$$\hat{P}_{n<n+l}(\alpha) \equiv |n<n+l;\hat{\alpha}\rangle \langle n<n+l;\hat{\alpha}| \ , \tag{4.27}$$

the state $|n<n+l;\hat{\alpha}\rangle$ being given by eq. (4.21).

It is crucial to gather that the action of the lowering and raising degeneracy operators $\theta(\alpha), \bar{\theta}(\alpha)$ can be extended on the singular states in terms of their L^2-expansions: namely,

$$\theta(\alpha)\,|n \geq n-l;\check{\alpha}\rangle \equiv \sum_{k=0}^{\infty} \check{c}_{n,l}^{\,k} \ \theta(\alpha)\,|k \geq k-l;\hat{\alpha}\rangle =$$

$$\sum_{k=1}^{\infty} \check{c}_{n,l}^{\,k} \sqrt{k-l}\,|k > k-l-1;\hat{\alpha}\rangle = \sqrt{n-l+\alpha}\,|n > n-l-1;\check{\alpha}\rangle \ , \tag{4.28a}$$

$$\bar{\theta}(\alpha)\,|n \geq n-l;\check{\alpha}\rangle \equiv \sum_{k=0}^{\infty} \check{c}_{n,l}^{\,k} \ \bar{\theta}(\alpha)\,|k \geq k-l;\hat{\alpha}\rangle =$$

$$\sum_{k=0}^{\infty} \check{c}_{n,l}^{\,k} \sqrt{k-l+1}\, |k > k-l+1; \hat{\alpha}\rangle = \sqrt{n-l+1+\alpha}\, |n \geq n-l+1; \check{\alpha}\rangle \ , \quad (4.28b)$$

$$n+1 \in \mathbf{N} \ , \qquad l = 0, 1, \ldots, n \ ;$$

$$\theta(\alpha)\, |n < n+l; \hat{\alpha}\rangle \equiv \sum_{k=0}^{\infty} \hat{c}_{n,l}^{\,k} \ \theta(\alpha)\, |k < k+l; \check{\alpha}\rangle =$$

$$\sum_{k=0}^{\infty} \hat{c}_{n,l}^{\,k} \sqrt{k+l+\alpha}\, |k < k+l-1; \check{\alpha}\rangle = \sqrt{n+l}\, |n < n+l-1; \hat{\alpha}\rangle \ , \qquad (4.29a)$$

$$\bar{\theta}(\alpha)\, |n < n+l; \hat{\alpha}\rangle \equiv \sum_{k=0}^{\infty} \hat{c}_{n,l}^{\,k} \ \bar{\theta}(\alpha)\, |k < k+l; \check{\alpha}\rangle =$$

$$\sum_{k=0}^{\infty} \hat{c}_{n,l}^{\,k} \sqrt{k+l+1+\alpha}\, |k < k+l+1; \check{\alpha}\rangle = \sqrt{n+l+1}\, |n < n+l+1; \hat{\alpha}\rangle \ , \quad (4.29b)$$

$$n+1 \in \mathbf{N} \ , \qquad l \in \mathbf{N} \ .$$

It should be noticed that the state vectors $\{|n \geq 0; \check{\alpha}\rangle \ , \ n+1 \in \mathbf{N}\}$, *i.e.*, the singular states of lowest occupation number corresponding to $n = l$, are such that

$$\theta(\alpha)\, |n \geq 0; \check{\alpha}\rangle \neq 0 \ , \qquad n+1 \in \mathbf{N} \ , \qquad (4.30)$$

which means that they are not annihilated by the lowering degeneracy operators, though giving rise to new state vectors of angular momentum $\ell = -\hbar(n+1)$ respectively. On the contrary, see eq. (3.16a), the regular state vectors $\{|n \geq 0; \hat{\alpha}\rangle \ , \ n+1 \in \mathbf{N}\}$ enjoy the property

$$\theta(\alpha)\, |n \geq 0; \hat{\alpha}\rangle = 0 \ , \qquad n+1 \in \mathbf{N} \ , \qquad (4.31)$$

whose role will be crucial in setting up the exact solution in the presence of the uniform electric field, as we shall see below.

Concerning self-adjoint extensions, we should go a little bit further in order to reach the most general statement. Actually, it can be proved that, for any fixed value $\ell = \hbar l, \ l \in \mathbf{Z}$ of the angular momentum, there is a continuous family of self-adjoint extensions of the radial Hamiltonians which interpolates between the regular one $h_l(\alpha)$ and the singular one

$H_l(\alpha)$. However, since none of those further possible self-adjoint extensions will be relevant in searching an exact solution of the one impurity problem in the presence of the uniform electric field, we shall no longer discuss here that quite interesting matter, but leave it to a forthcoming analysis. To sum up, we can say that the solution we have discussed in the previous section - in which the domain of the quantum Hamiltonian is that of the regular wave functions - actually corresponds to one-impurity described by a pure AB interaction. The further possible choices of the quantum self-adjoint Hamiltonians - such that the domains contain singular wave functions at the vortex position - do physically represent the simultaneous presence of the AB and contact- interactions.

5. Exact solution for the one-impurity quantum Hall's problem.

We are now ready to discuss the exact solution in the presence of the AB-vortex - the one-impurity problem - and of uniform electromagnetic fields. According to the conventional picture [2],[3],[8], it is conjectured that the presence of a not too large number of localized impurities within the Hall's sample, is actually what is needed to give account for the onset of the Hall's plateaus. To this respect, it is plausibly believed that the structure in terms of Landau's bands is basically kept, even in the presence of a small number of localized impurities, although the density of the states among and within the Landau's subbands is significantly changed by the presence of impurities (Landau's subbands are broadened and depopulated). It is commonly accepted that the above pattern eventually supports, in terms of various analytical approximate methods of investigations [2], some reasonable explanation for the Hall's plateaus. The basic idea behind this picture is that the switching on of a weak electric field is a smooth perturbation, whose net effect is, one the one hand, to lift the degeneracy of the conducting depopulated Landau's bands whereas, on the other hand, it does not prevent the presence of non-conducting localized eigenstates.

On the contrary, as we shall see below, the exact solution of the present model shows that the switching on of the uniform Hall field E_H drastically modify the distribution and the nature of the energy eigenstates, with respect to the situation in the absence of E_H and no matter how weak the Hall field is. In particular, all the energy eigenstates do contribute to the Hall current and the improper (*i.e.* extended) wave functions of the eigenstates necessarily become singular at the impurity position.

In order to prove the above statements, we shall solve our problem following a constructive approach, which makes use of all the detailed explicit information we have learned in the previous sections. As a matter of fact, what we have seen in the previous section is that the presence of the one-impurity AB-vortex might indeed realize what is largely believed: the integer valued Landau's levels are kept and, in general, further non-integer valued energy levels do actually appear, in such a way that the I.V.E. eigenstates degeneracy of the Landau's band is lowered, albeit still infinite. This means, in turn, that the density of the states is actually sensibly modified by the presence of the impurity. Consequently, what is reasonably expected - following the afore mentioned popular belief - after switching on some weak uniform electric Hall field, is that the conductance of the remaining charged states within the Landau's bands is slightly increased - with respect to the impurity-free case - in such a way that the net result for the Hall's conductance is again the "classical" one of eq. (2.30). As we will see below, it turns out that the exact solution actually suggests some quite different picture.

It is not difficult to verify that the rescaled hamiltonian differential operator, in the presence of an additional uniform electric field suitably described by the afore introduced parameter ϱ - see eq. (2.6) - can be written in the form

$$\frac{2m\lambda_B^2}{\hbar^2} H(\alpha, E_H) \equiv \mathtt{h}(\alpha, \varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha) + i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)) - \frac{3}{4}\varrho^2 \ , \tag{5.1}$$

in which the translated energy and degeneracy creation-annihilation operators are referred to be respectively

$$\delta_\varrho(\alpha) \equiv i\sqrt{2}\left\{\partial_{\bar{z}} + \frac{z}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\} = \bar{\delta}_\varrho^\dagger(\alpha) \ , \tag{5.2a}$$

$$\bar{\delta}_\varrho(\alpha) \equiv i\sqrt{2}\left\{\partial_z - \frac{\bar{z}}{4}\left(1 - \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\} = \delta_\varrho^\dagger(\alpha) \ , \tag{5.2b}$$

$$\theta_\varrho(\alpha) \equiv -i\sqrt{2}\left\{\partial_z + \frac{\bar{z}}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) - \frac{\varrho}{4}\right\} = \bar{\theta}_\varrho^\dagger(\alpha) \ , \tag{5.2c}$$

$$\bar{\theta}_\varrho(\alpha) \equiv -i\sqrt{2}\left\{\partial_{\bar{z}} - \frac{z}{4}\left(1 + \frac{\alpha}{[\gamma]}\right) + \frac{\varrho}{4}\right\} = \theta_\varrho^\dagger(\alpha) \ . \tag{5.2d}$$

Again, one can readily check that the following canonical commutation relations hold true: namely,

$$[\delta_\varrho(\alpha), \bar{\delta}_\varrho(\alpha)] = [\theta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 1 \ , \quad [\delta_\varrho(\alpha), \theta_\varrho(\alpha)] = [\delta_\varrho(\alpha), \bar{\theta}_\varrho(\alpha)] = 0 \ . \tag{5.3}$$

Now, owing to the above algebra, we have that the full hamiltonian differential operator $\mathtt{h}(\alpha, \varrho)$ and the translated "Landau-like" differential operator $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$ indeed commute, *i.e.*,

$$\left[\mathtt{h}(\alpha, \varrho), \bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)\right] = 0 \ . \tag{5.4}$$

It is important to gather that, unless we specify the domains of the above mentioned hamiltonian differential operators, they are only hermitean. Since we have to deal with well defined self-adjoint quantum Hamiltonians, we have to specify the (common) domain in which the commutation relation (5.4) still holds for the corresponding quantum self-adjoint Hamiltonians. But then, the fundamental theorem states that the self-adjoint realizations of the full rescaled Hamiltonian and of the translated "Landau-like" Hamiltonian must have a complete orthonormal set of common eigenstates.

First we prove that there is only a single choice of the domain of the quantum Hamiltonians which allows for a solution of the problem. As a matter of fact, it appears that the spectrum of any self-adjoint extension $\mathtt{H}(\alpha,\varrho)$ of the full rescaled Hamiltonian (5.1) is continuous owing to the presence of the operator

$$\mathtt{T}(\alpha,\varrho) \equiv i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)) - \frac{3}{4}\varrho^2 \ , \tag{5.5}$$

whose spectrum is manifestly continuos (in the "classical" impurity-free case it drives to the electric splitting of the Landau's degeneracy, see eq. (2.12)). Consequently the eigenfunctions of $\mathtt{H}(\alpha,\varrho)$ will be improper state vectors and must be, owing to $[\mathtt{H}(\alpha,\varrho), \Delta_L(\alpha,\varrho)] = 0$, common eigenstates of the corresponding self-adjoint extension $\Delta_L(\alpha,\varrho)$ of the translated "Landau-like" Hamiltonian $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$, whose spectrum is instead purely discrete. As a consequence, discrete energy levels of $\Delta_L(\alpha,\varrho)$ with a finite degeneracy are forbidden, cause the degenerate states are proper state vectors and a finite combination of them cannot produce an improper state. This means, in particular, that if we choose the domain to be, *e.g.*, that of the regular wave functions on the plane, then the full rescaled Hamiltonian (5.1) is not a self-adjoint operator.

This quite general and rigorous result is such a stringent constraint that we are left with only two possible options in order to obtain a solution, namely we have to investigate the self-adjoint extensions of the translated "Landau-like" Hamiltonian whose spectra are given by either non-integer rescaled eigenvalues $\hat{\varepsilon}_n = 2(n+|\alpha|),\ n+1 \in \mathbf{N}$, or, alternatively, by integer rescaled eigenvalues $\check{\varepsilon}_n = 2n,\ n+1 \in \mathbf{N}$.

In the former case, the self-adjoint translated "Landau-like" Hamiltonian is given by its spectral decomposition: namely,

$$\hat{\Delta}(\alpha,\varrho) \equiv \sum_{n=0}^{\infty} 2(n+|\alpha|) \left\{ \sum_{k=0}^{n} \hat{P}_{n\geq k}(\alpha,\varrho) + \sum_{k=n+1}^{\infty} \hat{P}_{n<k}(\alpha,\varrho) \right\} \ , \tag{5.6}$$

where the projectors onto translated regular and singular states are given respectively by

$$\hat{P}_{n\geq k}(\alpha,\varrho) \equiv |n \geq k; \hat{\alpha},\varrho\rangle \langle n \geq k; \hat{\alpha},\varrho| \ , \tag{5.7a}$$

$$\hat{P}_{n<k}(\alpha,\varrho) \equiv |n < k; \hat{\alpha},\varrho\rangle \langle n < k; \hat{\alpha},\varrho| \ . \tag{5.7b}$$

The explicit form of the above eigenstates, normalized to unity, is provided according to the general construction described in the previous section - see *Lemmas* ($\pm$) - *i.e.*,

$$|n,k;\hat{\alpha},\varrho\rangle = \sqrt{\frac{\Gamma(1-\alpha)}{(k!)\Gamma(n+1-\alpha)}} [\bar{\delta}_\varrho(\alpha)]^n [\bar{\theta}_\varrho(\alpha)]^k \, |0,0;\hat{\alpha},\varrho\rangle \ , \quad n+1, k+1 \in \mathbf{N} \ , \tag{5.8}$$

the holomorphic representation of the cyclic ground state being

$$\langle z\bar{z}|0,0;\hat{\alpha},\varrho\rangle = \frac{\gamma^{|\alpha|/2}\exp\{-(1/4)(z-\varrho)(\bar{z}-\varrho)\}}{\sqrt{2\pi\Gamma(1-\alpha)}} \frac{\exp\{-\varrho^2/4\}}{\sqrt{{}_1F_1(1-\alpha,1;\varrho^2/2)}} \ , \tag{5.9}$$

as it arises after generalization of eq.s (3.16). Notice that, within the eigenstates (5.8), the regular ones correspond to negative or vanishing angular momenta ($n \geq k$), whilst the singular ones to positive angular momenta ($n < k$). Furthermore, it is manifest from the spectral decomposition (5.6) that the quantum number k labels the infinite discrete Landau's degeneracy. It can be readily verified, taking the construction leading to eq.s (4.29) suitably into account, that the following properties hold true: namely,

$$\theta_\varrho(\alpha) \, |n,k;\hat{\alpha},\varrho\rangle = \sqrt{k} \, |n,k-1;\hat{\alpha},\varrho\rangle \ , \tag{5.10a}$$

$$\bar{\theta}_\varrho(\alpha) \, |n,k;\hat{\alpha},\varrho\rangle = \sqrt{k+1} \, |n,k+1;\hat{\alpha},\varrho\rangle \ , \tag{5.10b}$$

Now, in order to find the solution of the eigenvalue problem for the quantum self-adjoint Hamiltonian

$$\hat{H}(\alpha,\varrho) = \hat{\Delta}(\alpha,\varrho) + \mathtt{T}(\alpha,\varrho) \ , \tag{5.11}$$

let us consider the states

$$
\begin{aligned}
|n, p_\perp; \hat{\alpha}, \varrho\rangle &\equiv \sum_{k=0}^{\infty} c_k^{(n)}(\tilde{p})\, |n, k; \hat{\alpha}, \varrho\rangle \ , \\
\tilde{p} &\equiv p_\perp - \frac{1}{2}\varrho \ , \quad p_\perp \in \mathbf{R} \ ,
\end{aligned}
\tag{5.12}
$$

which are build up in close analogy with the "classical" solution (2.17) - with $c_k^{(n)}(\tilde{p})$ given by eq. (2.20) - and belong by definition to the continuous spectrum. Notice that, by construction, the above states are obviously eigenstates of the self-adjoint operator (5.6). It is not difficult to verify that

$$
\hat{H}(\alpha, \varrho)\, |n, p_\perp; \hat{\alpha}, \varrho\rangle = \left(2n - 2\alpha - \varrho p_\perp - \frac{1}{4}\varrho^2\right) |n, p_\perp; \hat{\alpha}, \varrho\rangle \ . \tag{5.13}
$$

It is worthwhile to remark that the key point to obtain the above result is the fact that the set of states (5.8) is closed with respect to the free action of the translated degeneracy operators - see eq.s (5.10). This crucial feature is peculiar of the set (5.8) and, in particular, does not keep true for the other complete orthonormal set $|n, k; \check{\alpha}, \varrho\rangle$ - see eq. (4.30) - which characterizes the self-adjoint extension of the Hamiltonian with only integer eigenvalues (like in the "classical" case). This is why the quantum Hamiltonian (5.11) is the only (essentially) self-adjoint operator with a continuous non-degenerate spectrum and which commutes with the Landau's like self-adjoint operator (5.6), *i.e.* the unique solution of our problem.

Now, it can be readily checked that the conductance does not change with respect to the "classical" impurity-free case [7]. As a matter of fact, starting again from the definition of the current operator

$$
\hat{J}_{1,2} = \frac{|e|\hbar}{m\lambda_B} \hat{P}_{1,2}(\alpha) \ , \tag{5.14}
$$

in which

$$
\hat{P}_1(\alpha) = -\frac{1}{\sqrt{2}} \left(\delta_\varrho(\alpha) + \bar{\delta}_\varrho(\alpha)\right) \ , \tag{5.15a}
$$

$$\hat{P}_2(\alpha) = \frac{i}{\sqrt{2}}\left(\delta_\varrho(\alpha) - \bar{\delta}_\varrho(\alpha) + \frac{i}{\sqrt{2}}\varrho\right) \ , \tag{5.15b}$$

it immediately follows that, for any normalizable wave packet

$$|n,[f];\hat{\alpha},\varrho\rangle = \int_{-\infty}^{+\infty} dp_\perp \ f(p_\perp)\,|n,p_\perp;\hat{\alpha},\varrho\rangle \ , \quad \int_{-\infty}^{+\infty} dp_\perp \,|f(p_\perp)|^2 = 1 \ , \tag{5.16}$$

which describes one electron in the n-th conducting band, we obtain once again that the average current carried by such a state is

$$\left\langle n,[f];\hat{\alpha},\varrho|\hat{J}_1|n,[f];\hat{\alpha},\varrho\right\rangle = 0 \ , \tag{5.17a}$$

$$\left\langle n,[f];\hat{\alpha},\varrho|\hat{J}_2|n,[f];\hat{\alpha},\varrho\right\rangle = -|e|c\frac{E_H}{B} \ , \tag{5.17b}$$

which shows that the Hall's conductance is always the "classical" one as in eq. (2.30), even in the presence of the AB-vortex. A further important remark is now in order. Taking the limit when α is going to zero of the self-adjoint Hamiltonian (5.11) we recover the standard impurity-free self-adjoint Hamiltonian (2.6), whose domain is that of regular wave functions. This means that, if we eliminate the Aharonov-Bohm vortex potential, contact-interaction is no longer allowed in the presence of a uniform electromagnetic field. Consequently, the analysis of the self-adjoint extensions of the quantum Hamiltonian does contradict the claim in [3]. It should also be remarked that the switching on of a weak Hall electric field does represent a small and smooth perturbation on the system, only when the starting unperturbed Hamiltonian is

$$\hat{\Delta}(\alpha) \equiv \sum_{n=0}^{\infty} 2(n+|\alpha|)\left\{\sum_{k=0}^{n}\hat{P}_{n\geq k}(\alpha) + \sum_{k=n+1}^{\infty}\hat{P}_{n<k}(\alpha)\right\} \ . \tag{5.18}$$

Otherwise, for any different choice of the unperturbed Hamiltonian, the additional electric field cannot represent a smooth perturbation, since it involves a change in the domain of the Hamiltonian.

6. Conclusion.

In this paper we have explicitely solved the quantum mechanical *2+1* dimensional problem of the non-relativistic electron in the presence of a uniform electromagnetic field and of an Aharonov-Bohm vortex potential. The solution is unique, since it turns out that the quantum Hamiltonian is essentially self-adjoint in the presence of the uniform electric field. This is no longer true in the absence of the electric field: in the latter case - under the assumption of the $O(2)$-symmetry - each radial Hamiltonian allows for a one-parameter family of self-adjoint extensions. From the explicit knowledge of the eigenvalue and eigenfunctions of the full Hamiltonian, it is possible to compute the current and conductance, the results being the very same as in the "classical" case, *i.e.*, in the absence of the AB-vortex. It should be emphasized that the possibility to obtain the exact non-perturbative solution is heavily supported by the systematic application of the algebraic method, which allows to overcome the conflict between the rotational symmetry in the absence of the electric field and the explicit symmetry breaking due to the switching on of the uniform electric field itself.

The final result is that the Hall conductance does not change in the presence of one impurity described by the AB-vortex. The microscopic picture which emerges from the exact solution of the present one-impurity model can be summarized as follows. In the absence of the electric field, the general pattern can be described, as it was basically known [9], by the presence of depopulated integer-valued Landau's levels and of further real-valued energy levels of finite degeneracy - the details of this description depending upon the specific self-adjoint extension of the quantum Hamiltonian, as it was carefully explained in section 3.

On the ground of this model, one is led to the concept of the "broadening" of the

Landau's subbands, owing to the presence of impurities, and to the hope that the switching on of a weak electric field does basically keep this feature: the Hall conducting states of the electrons should be only those ones lying within the integer-valued Landau's subbands, the remaining allowed states giving no contribution to the Hall's conductivity. On the contrary, the exact solution of the present simple model shows that the switching on of the uniform electric field drastically changes the above picture: the electrically splitted Landau's levels are shifted with respect to the impurity-free case - see eq.s (2.18) and (5.13) - all the energy eigenstates belong to the electrically splitted Landau's subbands and the quantum eigenstates within each subband are necessarily described by a singular wave function - see section 5 - at variance with the impurity-free case. It is quite remarkable that, in spite of the above drastic reshuffling of the quantum states after the switching on of the electric field, the current and conductance are exactly the same with and without the AB-vortex. This fact appears to corroborate some deep topological nature of the Hall's conductance, as it is widely recognized in the literature [2]. As a matter of fact, the unavoidable presence of singularities of the eigenfunctions at the vortex position clearly represents the token of topological non-triviality. In other words, since the exact solution of the one-impurity problem yet necessarily involves singular wave functions at the impurity position, it means that the underlying configuration manifold in the general dynamical problem is the punctured plane, which is topologically non-trivial. This is *a fortiori* true in the realistic many impurities problem, whose exact solution is still unknown.

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Appendix

Here we want to explicitely verify eq. (4.13): namely

$$\sum_{k=0}^{\infty} \left|\check{c}_{n,l}^{\,k}\right|^2 = \frac{\{(n+\alpha)(\alpha)_n\}^2}{n!\Gamma(1-\alpha)\Gamma(n-l+\alpha+1)[(-n-\alpha)_l]^2} \sum_{k=0}^{\infty} \frac{(1-\alpha)_k}{(k-l)!(n+\alpha-k)^2} \tag{A1}$$
$$= 1 \ , \qquad l = 1, 2, \ldots, n \ .$$

To this aim, let us first consider the convergent series

$$\sigma_\nu^{(l)}(\alpha) \equiv \sum_{k=0}^{\infty} \frac{(1-\alpha)_k}{(k-l)!(\alpha-k-\nu)^2} \ , \quad \nu \in \mathbf{C} \ , \tag{A2}$$
$$\nu \neq \alpha - k \ , \quad l = 1, 2, \ldots, n \ , \quad -1 < \alpha < 0 \ .$$

which precisely reproduces the series of eq. (A1) in the limit $\nu \to -n$. The last equation can be rewritten as

$$\sigma_\nu^{(l)}(\alpha) = \int_0^\infty dt \ t \exp\{-t(\nu + k - \alpha)\} \sum_{k=0}^{\infty} \frac{(1-\alpha)_k}{(k-l)!} \tag{A3}$$
$$= \lim_{z \uparrow 1} \sigma_\nu^{(l)}(\alpha; z) \ ,$$

where we have set

$$\sigma_\nu^{(l)}(\alpha; z) \equiv (-1)^{l+1} (\alpha-1)(\alpha-2)\ldots(\alpha-l) \int_0^1 dx \ x^{\nu-\alpha+l-1} (1-zx)^{\alpha-l-1} \ln x \ . \tag{A4}$$

Explicit evaluation yields [14]

$$\lim_{z \uparrow 1} \sigma_\nu^{(l)}(\alpha; z) = (1-\alpha)_l B(\nu - \alpha + l, \alpha - l) \{\psi(\nu) - \psi(\nu - \alpha + l)\} \ . \tag{A5}$$

Taking eventually the limit $\nu \to -n$ we get

$$\sigma_{-n}^{(l)}(\alpha) = \frac{\pi n!}{\sin \pi\alpha} \frac{\Gamma(\alpha - l)}{\Gamma(n+1+\alpha)} (1-\alpha)_l (-n-\alpha)_l \ , \tag{A6}$$

which readily proves eq. (A1), as promised. In a quite similar way, it is not difficult to verify eq. (4.23) too, what concludes our proof.

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