Krein's formula in the presence of Gravity, Floor and Contact Interaction

• 2ω spatial dimensions:

$$\mathbf{r} = (\mathbf{r}_\perp, x_{2\omega} \equiv x) \ , \quad \mathbf{r}_\perp = (x_1, x_2, \ldots, x_{2\omega-1}) \ ; \tag{1}$$

$$\mathbf{p} = (\mathbf{p}_\perp, p_{2\omega} \equiv p) \ , \quad \mathbf{p}_\perp = (p_1, p_2, \ldots, p_{2\omega-1}) \ ; \tag{2}$$

the Floor is at $x = -L$ and we set

$$\ell_g \equiv \kappa^{-1} = \left(\frac{\hbar^2}{2m^2 g} \right)^{1/3} \ , \tag{3}$$

$$E_g \equiv \frac{mg}{\kappa} = \frac{\hbar^2 \kappa^2}{2m} \ . \tag{4}$$

• Hamiltonian

$$\begin{aligned} H(g, L, \lambda) &= \frac{\mathbf{p}^2}{2m} + mgx + \lambda \delta^{(2\omega)}(\mathbf{r}) \\ &\equiv H_0(g, L) + \lambda \delta^{(2\omega)}(\mathbf{r}) \ , \\ \mathbf{r} \in \mathbf{R}_+^{2\omega} &\equiv \left\{ \mathbf{r} = (\mathbf{r}_\perp, x) | \mathbf{r}_\perp \in \mathbf{R}^{2\omega-1}, \ x \geq -L \right\} \ . \end{aligned} \tag{5}$$

• Eigenfunctions and eigenvalues of $H_0(g, L)$

$$\Psi_{n,\mathbf{p}_\perp}(\mathbf{r}) = \frac{\sqrt{\kappa}}{h^{\omega - \frac{1}{2}} \mathrm{Ai}'(a_n)} \ \exp \left\{ \frac{i}{\hbar} \mathbf{p}_\perp \cdot \mathbf{r}_\perp \right\} \mathrm{Ai}\left(\kappa x + \kappa L + a_n \right) \ , \tag{6}$$

$$E_{n,\mathbf{p}_\perp} = \frac{\mathbf{p}_\perp^2}{2m} - E_g(\kappa L + a_n) \ , \qquad n + 1 \in \mathbf{N} \ , \quad \mathbf{p}_\perp \in \mathbf{R}^{2\omega-1} \ , \tag{7}$$

where a_n are the zeroes of the Airy's functions, *i.e.* $\mathrm{Ai}(a_n) = 0$, whilst Dirichelet's boundary conditions are imposed on the Floor, namely:

$$\psi(\mathbf{r}_\perp, x = -L) = 0 \ , \quad \forall \psi \in L^2(\mathbf{R}_+^{2\omega}) \ . \tag{8}$$

The spectrum is purely continuous iff $D \neq 1 \Leftrightarrow \omega \neq \frac{1}{2}$ and the corresponding improper eigenfunctions are normalized according to

$$\left\langle \Psi_{n',\mathbf{p}'_\perp} | \Psi_{n,\mathbf{p}_\perp} \right\rangle = \delta_{n,n'} \ \delta^{(2\omega-1)}(\mathbf{p}_\perp - \mathbf{p}'_\perp) \ . \tag{9}$$

• Green's function of the Hamiltonian $H_0(g,L)$:

$$G_0\left(z,g,L;\mathbf{r},\mathbf{r}'\right) = i\ \frac{\kappa m}{2\hbar^2}\ \left(2\pi\rho_\perp\right)^{\frac{3}{2}-\omega}\times$$

$$\sum_{n=0}^{\infty}\frac{q_n^{\omega-\frac{3}{2}}}{\mathrm{Ai}'^2(a_n)}H^{(1)}_{\omega-\frac{3}{2}}\left(q_n\rho_\perp\right)\mathrm{Ai}\left(\kappa x+\kappa L+a_n\right)\mathrm{Ai}\left(\kappa x'+\kappa L+a_n\right)\ , \tag{10}$$

in which we have set

$$\hbar q_n \equiv \sqrt{2mz+2mE_g(\kappa L+a_n)}\ , \quad \rho_\perp \equiv \sqrt{(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}\ . \tag{11}$$

• Asymptotic behaviours $n \gg 1$ [AS]:

$$a_n = -f\left[\frac{3\pi}{8}(4n+3)\right] \sim -\left(\frac{3\pi}{2}n\right)^{\frac{2}{3}}\ , \tag{12}$$

$$\mathrm{Ai}^2\left(\kappa L+a_n\right) \sim \frac{1}{2\pi}\left(\frac{3\pi}{2}n\right)^{-\frac{1}{3}}\ , \tag{13}$$

$$\mathrm{Ai}'^2\left(a_n\right) = f_1^2\left[\frac{3\pi}{8}(4n+3)\right] \sim \frac{1}{\pi}\left(\frac{3\pi}{2}n\right)^{\frac{1}{3}}\ , \tag{14}$$

$$A_n \equiv \frac{\mathrm{Ai}^2\left(\kappa L+a_n\right)}{\mathrm{Ai}'^2\left(a_n\right)} \sim \frac{1}{2}\left(\frac{3\pi}{2}n\right)^{-\frac{2}{3}}\ . \tag{15}$$

• Renormalization and bound states: the denominator of the Krein's formula reads

$$\frac{1}{\lambda}+G_0\left(z,g,L;\mathbf{0},\mathbf{0}\right) = \frac{1}{\lambda_R}+G_0^R\left(z,g,L;\mathbf{0},\mathbf{0}\right)\ . \tag{16}$$

The zeroes of the above renormalized ($D=2 \Leftrightarrow \omega=1$ and $D=3 \Leftrightarrow \omega=\frac{3}{2}$) denominators will provide poles in the resolvent $G\left(z,g,L,\lambda;\mathbf{r},\mathbf{r}'\right)$ corresponding to the possible existence of bound states owing to the presence of Contact Interaction. In order to answer this question one should, first, sum up the series

$$\sum_{n=0}^{\infty}\frac{H^{(1)}_0\left(q_n\rho_\perp\right)}{\mathrm{Ai}'^2(a_n)}\mathrm{Ai}\left(\kappa x+\kappa L+a_n\right)\mathrm{Ai}\left(\kappa x'+\kappa L+a_n\right) =$$

$$\frac{2\hbar^2}{i\kappa m}\ G_0^{(3D)}(z,g,L;\mathbf{r},\mathbf{r}')\ , \tag{17}$$

$$\sum_{n=0}^{\infty}\frac{H^{(1)}_{-\frac{1}{2}}\left(q_n\rho_\perp\right)}{\sqrt{q_n}\mathrm{Ai}'^2(a_n)}\mathrm{Ai}\left(\kappa x+\kappa L+a_n\right)\mathrm{Ai}\left(\kappa x'+\kappa L+a_n\right) =$$

$$\frac{\hbar^2}{i\kappa m}\sqrt{\frac{2}{\pi\rho_\perp}}\ G_0^{(2D)}(z,g,L;\mathbf{r},\mathbf{r}')\ , \tag{18}$$

then subtract the ultraviolet divergencies at $\mathbf{r} = \mathbf{r}' = 0$ and finally solve with respect to z the above basic equation (16). This program is very difficult to be done analitically, maybe it can be approached numerically with success: at the moment we shall give below asymptotic estimates in the continuum limit that suggest the existence of at least one bound state, what is enough to allow Bose-Einstein Condensation for an ideal gas of those particles in such external potentials.

• Asymptotic estimates in the continuum limit: when n is sufficiently large, the nth term of the above series can be approximated with the asymptotic formulae listed above and the series can be substituted with integrals, as the zeroes of the Airy's functions become more and more dense. In so doing one obtains

$$G_0^{(3D)}(z,g,L;\rho_\perp,x=x'=0) \sim \frac{2\pi m}{\rho_\perp h^2}\exp\left\{-\frac{\rho_\perp}{\hbar}\sqrt{-2mz}\right\}\ , \tag{19}$$

$$G_0^{(2D)}(z,g,L;\rho_\perp,x=x'=0) \sim \frac{im}{2\hbar^2}H_0^{(1)}\left(\frac{\rho_\perp}{\hbar}\sqrt{2mz}\right)\ , \tag{20}$$

which nicely correspond to resolvents of the free particles in the absence of Gravity and Floor. Renormalization is done accordingly in the standard way: the conclusion is that, in the continuum limit, the denominators of the Krein's formaulae in three and two dimensions admit at least one zero and bound states therefore seems to exist. In other words, it seems that the presence of the Floor renders stable the (lowest) bound state which is only metastable in the absence of the Floor. Of course, these considerations are quite heuristic and qualitative although they could appear physically reasonable.

• Integral representation: on the other hand, the Green's function can also be expressed in terms of the following integral representation (according to eq. (10) of your notes and in physical units)

$$\begin{aligned} G_0\left(z,g,L;\mathbf{0},\mathbf{0}\right) = \frac{\pi\kappa}{E_g h^{2\omega-1}}\int d^{2\omega-1}p_\perp\ \mathrm{Ai}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}\right)\times \\ \left\{\mathrm{Bi}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}\right)-\mathrm{Ai}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}\right)\frac{\mathrm{Bi}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}-\kappa L\right)}{\mathrm{Ai}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}-\kappa L\right)}\right\}\ . \end{aligned} \tag{21}$$

It appears to be not easy, at least at first sight, to extract from the above integral representation the eventual divergent parts in 2D and 3D as well as to find the solutions of the basic equation

$$\frac{1}{\lambda_R} + G_0^R\left(z,g,L;\mathbf{0},\mathbf{0}\right) = 0\ , \tag{22}$$

which, in particular, would provide at least the lowest discrete energy level that eventually would allow the BEC.

• 3D case: here we can solve the first integral in eq. (21) thanks to the indefinite integral

$$\int X(x)Y(x)dx = xX(x)Y(x) - X'(x)Y'(x) \ , \tag{23}$$

where X, Y are solutions of the Airy's equation. Then we find

$$\begin{aligned}
&\frac{\pi\kappa}{E_g h^2}\int d^2p_\perp \ \mathrm{Ai}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}\right)\mathrm{Bi}\left(\frac{p_\perp^2}{2mE_g}-\frac{E}{E_g}\right) = \\
&\frac{m\kappa}{2\hbar^2}\lim_{X\to\infty}\int_{-(E/E_g)}^{X} dx \ \mathrm{Ai}(x)\mathrm{Bi}(x) = \\
&\frac{m\kappa}{2\hbar^2}\lim_{X\to\infty}\left\{\frac{\sqrt{X}}{\pi}+\frac{E}{E_g}\mathrm{Ai}\left(-\frac{E}{E_g}\right)\mathrm{Bi}\left(-\frac{E}{E_g}\right)+\mathrm{Ai}'\left(-\frac{E}{E_g}\right)\mathrm{Bi}'\left(-\frac{E}{E_g}\right)\right\} \ ,
\end{aligned} \tag{24}$$

a result that clearly needs renormalization, as expected, and suggests an infinite number of discrete positive energy solutions of the basic equation (22), although evaluation of the second integral is necessary but harder.

• 1D case: in the one dimensional case the spectrum is purely discrete and, generally speaking, the Krein's formula does not hold true as it concerns the case of a continuous spectrum. The purely discrete spectrum of the one-dimensional case can be implicitely obtained after solving directly the Schroedinger equation in the tempered distribution sense: namely,

$$\left\{-\frac{\hbar^2}{2m}\mathrm{d}_x^2 + mgx + \lambda\delta(x) - E\right\}\psi_E(x) = 0 \ . \tag{25}$$

After setting

$$\epsilon \equiv \frac{E}{E_g} \ , \qquad \alpha \equiv \frac{\kappa\lambda}{E_g} \ , \tag{26}$$

the solution of eq. (25) reads

$$\pi\alpha\mathrm{Ai}(-\epsilon)\mathrm{Bi}(-\epsilon-\kappa L)\left\{\frac{\mathrm{Bi}(-\epsilon)}{\mathrm{Bi}(-\epsilon-\kappa L)} - \frac{\mathrm{Ai}(-\epsilon)}{\mathrm{Ai}(-\epsilon-\kappa L)}\right\} = 1 \ , \tag{27}$$

that clearly leads to a purely discrete spectrum. It is quite curious that, after putting $\omega = \frac{1}{2}$ in eq. (21), the above relationship (27) - in the 1D case no renormalization is needed - precisely corresponds, as you can easily check, to

$$\frac{1}{\lambda} + G_0\left(z, g, L; \mathbf{0}, \mathbf{0}\right)\big|_{\omega=\frac{1}{2}} = 0 \ , \tag{28}$$

in spite of the fact that the Krein's formula can not be proved in the case of a purely discrete spectrum.

References

[AS] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972) 446-452.