BOSE-EINSTEIN CONDENSATION IN THE PRESENCE OF ONE IMPURITY AND A UNIFORM FIELD

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0. INTRODUCTION

1. POINT-LIKE IMPURITY

2. BOSE-EINSTEIN CONDENSATION

3. UNIFORM FIELD

4. CONCLUSIONS

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0. INTRODUCTION

- **Ideal Quantum Systems in** D **Spatial Dimensions in the presence of External Fields**

  ◦ Gravity-like Uniform Fields

  ◦ crossed Hall's Fields

  ◦ Point-like Disorder : *AB-vortex & δ-like Interaction*

- *δ-like Potentials* **are well defined only in** $D = 1$

  $D = 2, 3$ **Renormalization⇔Contact Interaction**

- **Solvable Mathematical Models**

  Self-Adjoint Exts of Symmetric Hamiltonians

  ◦ *von Neumann's Method of Deficiency Indices*

  ◦ *Krein's Formula for the Resolvent*

$$\Updownarrow$$

Contact Interaction

" ... If the particles of a 3D ideal boson gas were placed in a uniform gravitational field, then Bose-Einstein condensation would still occur, but in the condensation region there would be a spatial separation of the two phases, just as in a gas-liquid condensation ... " pg. 266

K. Huang, *Statistical Mechanics* (Wiley, New York, 1987)

long standing popular belief

L. Goldstein, J. Chem. Phys. **9**, 273 (1941)

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S. R. de Groot, G. J. Hooyman and C. A. ten Seldam, Proc. R. Soc. London A **203**, 266 (1950)

O. Halpern, Phys. Rev. **86**, 126 (1952); **87**, 520 (1952).

H. A. Gersch, J. Chem. Phys. **27**, 928 (1957)

V. Bagnato, D. E. Pritchard and D. Kleppner, Phys. Rev. A **35**, 4354 (1987)

etc.

Theorem

Suppose the one-particle energy spectrum of an ideal boson gas satisfies the following conditions

(i) *there is a gap between the fundamental and the first excited energy levels*

$$E_1 - E_0 = \Delta > 0$$

(ii) *the one-particle partition function is finite*

$$Z \equiv \sum_{n=0}^{\infty} d_n \exp(-\beta E_n) < \infty$$

d_n *being the finite degeneracy of the* n*-th eigenvalue of the one-particle Hamiltonian*

Then this gas displays Bose-Einstein condensation

`Proof`

If $\mu < E_0$ the No. of particles in the excited states is bounded

$$N_{\rm ex} = \sum_{n=1}^{\infty} \frac{d_n \exp[-\beta(E_n - \mu)]}{1 - \exp[-\beta(E_n - \mu)]}$$

$$\leq \frac{\exp(\beta\mu)}{1 - \exp[-\beta(E_1 - \mu)]} \sum_{n=1}^{\infty} d_n \exp(-\beta E_n)$$

Therefore

$$\lim_{\mu \to E_0} N_{\rm ex} \leq \frac{\exp(\beta E_0)}{1 - \exp(-\beta\Delta)} \left[Z - d_0 \exp(-\beta E_0)\right] < \infty$$

since by hypothesis $\Delta > 0$ and Z and d_0 are finite *QED*

generalization to

continuous spectrum & degeneracy

under the suitable introduction of

`1-particle partition function` *per* `unit volume`

`density of particles in the excited states`

1. POINT-LIKE IMPURITY

- Classical point-like impurity in 2D : AB-vortex

  (Aharonov & Bohm 1959)

$$H(\alpha) = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2$$

$$A_j(x_1, x_2) = \frac{\phi}{2\pi}\epsilon_{jk}\frac{x_k}{r^2}$$

$$\alpha = \frac{\phi}{\phi_0} = \frac{e\phi}{hc} \qquad \phi_0 \equiv \frac{hc}{e} \sim 10^{-4}\mathrm{Gauss} \times \mathrm{cm}^2$$

$$\mathbf{p}, \mathbf{r} \in \mathbf{R}^2 \qquad r \equiv \sqrt{x_1^2 + x_2^2}$$

- **single valuedness** $\Longrightarrow$ $-1 < \alpha \leq 0$

$*$ quantization of classical Hamiltonian $*$

$*$ leads to **SYMMETRIC** operator $*$

$$\Downarrow$$

FIND ALL THE SELF-ADJOINT EXTENSIONS

$$\Updownarrow$$

CONTACT INTERACTION (CI)

in QUANTUM MECHANICS

(*Fermi 1936*)

$\odot$ $D = 1$ **CI fully equivalent** to δ-potential

$\odot$ $D > 1$ **CI NOT equivalent** to δ-potential

(*Albeverio Gesztesy Høegh-Krohn Holden 1988*)

- **Search $O(2)$-invariant** Self-Adjoint Extensions

  solve the radial problem of angular momentum l

$\odot$ $$\psi_l(kr) = A_l J_{|l+\alpha|} + B_l N_{|l+\alpha|}$$

To fix A_l and B_l

$$\Updownarrow$$

impose boundary conditions

at the impurity position $r = 0$

$$\Updownarrow$$

Self-Adjointness of the Hamiltonian operator

- square integrability around impurity position

$$B_l = 0 \quad \Leftrightarrow \quad l \neq 0$$

$\odot$ Let us fix the S-**wave** coefficients A_0 and B_0

Continuous Spectrum

- Eigenvalues $E = (\hbar^2 k^2/2m) \qquad k > 0$

- Improper Eigenfunctions

$$\psi_l(k; r, \theta) = \sqrt{\frac{k}{2\pi}} e^{il\theta} J_{|l+\alpha|}(kr) \qquad l \in \mathbf{Z} - \{0\}$$

- $\psi_0(k, E_0; r) = A_0(k; \alpha, E_0) J_{|\alpha|}(kr) + B_0(k; \alpha, E_0) N_{|\alpha|}(kr)$

$\star$ $E_0 \in \mathbf{R}$ is some suitable **physical energy scale** $\star$

- Small distance behaviour

$$N_{|\alpha|}(kr) \sim \frac{\csc(\pi\alpha)}{\Gamma(1 - |\alpha|)} \left(\frac{kr}{2}\right)^{-|\alpha|} \qquad (r \sim 0)$$

- **Orthonormality**

$$\lim_{R\to\infty}\int_0^R 2\pi dr\; r\psi_0(k,E_0;r)\psi_0^*(k',E_0;r) = \delta(k-k')$$

$$\Updownarrow$$

$$\frac{B_0(k;\alpha,E_0)}{A_0(k;\alpha,E_0)} = \frac{\sin(\pi\alpha)}{\cos(\pi\alpha)+\mathrm{sgn}(E_0)\left(\hbar^2k^2/2m|E_0|\right)^{\alpha}}$$

- **Regular solution** $\Leftrightarrow$ Aharonov-Bohm wave function

$$B_0(k;\alpha,E_0)=0 \;\Leftrightarrow\; E_0\to\pm\infty$$

$$\Updownarrow$$

$\star$ **no Contact Interaction** $\star$

- **Special cases**

$|\alpha|=0$ **pure** Contact Interaction in 2D

$|\alpha|=\frac{1}{2}$ **pure** Contact Interaction in 3D

Discrete Spectrum

Completeness $\Leftrightarrow$ **one bound state exists iff** $E_0 < 0$

- $$\langle r|\psi_B\rangle = \psi_B(\kappa, r) = \frac{\kappa}{\pi}\sqrt{\frac{\sin(\pi\alpha)}{\alpha}}K_\alpha(\kappa r)$$

$$\hbar\kappa \equiv \sqrt{2m|E_0|}$$

$$\Updownarrow$$

Self-Adjoint $O(2)$-**invariant** Hamiltonians

defined by their **spectral decompositions**

- $$H(\alpha, E_0) = \sum_{l=-\infty}^{+\infty}\int_0^\infty dk \frac{\hbar^2 k^2}{2m} |l,k\rangle \langle k,l| + \vartheta(-E_0)E_0 |\psi_B\rangle \langle\psi_B|$$

- $$\langle r,\theta|l,k\rangle = \frac{\exp\{il\theta\}}{\sqrt{2\pi}}\psi_l(k,r;\alpha,E_0) \qquad k \geq 0$$

$\star$ REMARKS $\star$

- **Physical meaning** of the quantum energy scale E_0

  $E_0 < 0 \quad \Longrightarrow \quad$ **bound state energy**

  $E_0 > 0 \quad \Longrightarrow \quad$ **resonance energy**

$$\delta_l(\alpha) = \frac{\pi}{2}|\alpha|\mathrm{sgn}(l) = \frac{\pi}{2}\left[|l| - |l+\alpha|\right] \qquad l \neq 0$$

$$\delta_0(k;\alpha,E_0) = \frac{\pi}{2}\,\alpha - \mathrm{arctg}\left[\frac{\sin(\pi\alpha)}{\cos(\pi\alpha) + \mathrm{sgn}(E_0)(k^2/|E_0|)^{\alpha}}\right]$$

- $H(\alpha, E_0)$ is a **continuous one-parameter family** of Self-Adjoint Extensions of the symmetric Hamiltonian $H(\alpha)$ labelled by the physical quantum energy scale E_0 (von Neumann's theorem)

$\star$ FUNDAMENTAL THEOREM $\star$

Self-Adjointness

$$\Updownarrow$$

Orthonormality & Completeness

$$\Updownarrow$$

von Neumann's method of deficiency indices

Special cases

- $|\alpha| = 0$ **pure** Contact Interaction in 2D

$$H_{\rm 2D}(E_0) = \sum_{l=-\infty}^{+\infty} \int_0^{\infty} dk \frac{\hbar^2 k^2}{2m} |l,k\rangle \langle k,l| + E_0 |\psi_B\rangle \langle \psi_B| \qquad [E_0 < 0]$$

$\star$ **the bound state always exists** $\star$

- $|\alpha| = \frac{1}{2}$ **pure** Contact Interaction in 3D

$$H_{\rm 3D}(E_0) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \int_0^{\infty} dk \frac{\hbar^2 k^2}{2m} |l,m,k\rangle \langle k,m,l| + \vartheta(-E_0) E_0 |\psi_B\rangle \langle \psi_B| \qquad [E_0 \in \mathbf{R}]$$

$\star$ **dimensional transmutation** $\star$

$\star$ **Point-like DISORDER** $\star$

- If the **Domain** of the Hamiltonian operator is that of wave functions REGULAR on the hole plane

$$\Updownarrow$$

$$E_0 = \pm\infty \qquad H(\alpha, E_0 = \pm\infty) \equiv H_{\text{AB}}(\alpha)$$

absence of Contact Interaction

- If the **Domain** of the Hamiltonian operator is that of wave functions SINGULAR at the impurity position

$$\Updownarrow$$

$$E_0 \neq \pm\infty \qquad H = H(\alpha, E_0)$$

presence of Contact Interaction

$\odot$ point-like disorder in 2D is described in **QM** by **TWO** parameters α, E_0

2. BOSE-EINSTEIN CONDENSATION

- **To discuss BEC of ideal systems we need**

○ `the 1-particle partition function`

$$Z_{\rm 2D}(\alpha, \beta, E_0)$$

○ `the density of 1-particle states`

$$\rho_{\rm 2D}(\alpha, E, E_0)$$

○ **the average particles density**

$$\langle n(\alpha, \beta, \mu, E_0) \rangle_{\rm 2D}$$

$$\beta \equiv \frac{1}{k_B T} \qquad\qquad z \equiv e^{\beta\mu} \quad \text{fugacity}$$

The 1-particle partition function

in a large 2D box of area A

$$Z_{\rm 2D}(\alpha,\beta,E_0)=\frac{\rm A}{\lambda_T^2}+\frac{\alpha(\alpha+1)}{2}+\vartheta(-E_0)e^{\beta|E_0|}$$

$$+\frac{\alpha\sin(\pi\alpha)}{\pi}\int_0^\infty\frac{dx}{x^{1+\alpha}}\frac{{\rm sgn}(E_0)\ e^{-\beta|E_0|x}}{1+2{\rm sgn}(E_0)x^{|\alpha|}\cos(\pi\alpha)+x^{2|\alpha|}}$$

$$\lambda_T\equiv\frac{h}{\sqrt{2\pi mk_BT}}$$

Infrared Divergence A $\rightarrow\infty$

only in the free part

Special Cases

- $|\alpha| = 0$ **pure CI in 2D** $[E_0 < 0]$

$$Z_{2\mathrm{D}}(\beta, E_0) = \frac{\mathrm{A}}{\lambda_T^2} + e^{\beta|E_0|} - \int_0^\infty \frac{dE}{E} \frac{e^{-\beta E}}{\ln^2(-E/E_0) + \pi^2}$$
$$= \frac{\mathrm{A}}{\lambda_T^2} + \nu(\beta|E_0|)$$

$$\nu(x) \equiv \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt$$

the bound state is always present

- $|\alpha| = \frac{1}{2}$ **pure CI in 3D** $[E_0 \in \mathbf{R}]$

$$Z_{3\mathrm{D}}(\beta, E_0) = \frac{\mathrm{V}}{\lambda_T^3} + \vartheta(-E_0) e^{\beta|E_0|}$$

$$-\frac{1}{8} + \frac{1}{2}\mathrm{sgn}(E_0) e^{\beta|E_0|} \mathrm{erfc}(\sqrt{\beta|E_0|})$$

the bound state is present iff $E_0 < 0$

$\star$ **dimensional transmutation** $\star$

`The density of 1-particle states`

Laplace inverse transform of $Z_{\rm 2D}$

$$\rho_{\rm 2D}(\alpha, E, E_0) \;=\; \vartheta(E)\frac{2\pi m}{h^2}{\rm A} + \frac{\alpha(\alpha+1)}{2}\delta(E) + \varrho(\alpha, E, E_0)$$

$$\varrho(\alpha, E, E_0) \equiv \frac{\alpha\sin(\pi\alpha)}{\pi E}\;\vartheta(E){\rm sgn}(E_0) \times \frac{(E|E_0|)^{|\alpha|}}{E^{2|\alpha|} + |E_0|^{2|\alpha|} + 2{\rm sgn}(E_0)(E|E_0|)^{|\alpha|}\cos(\pi\alpha)}$$

`The mean density at thermal equilibrium`

$$\langle n(\alpha, \beta, \mu, E_0)\rangle_{\rm 2D} = \lambda_T^{-2} g_1(z) + \frac{z\alpha(\alpha+1)}{2{\rm A}(1-z)} + \vartheta(-E_0)\frac{z}{{\rm A}(z_0 - z)} + \vartheta(E_0)\frac{z}{{\rm A}(1-z)} + \frac{z}{\rm A}\int_0^\infty dE\;\varrho(\alpha, E, E_0)\frac{e^{-\beta E}}{1 - z\exp\{-\beta E\}}$$

$$z_0 \equiv \exp\{\beta E_0\} \qquad\qquad g_1(z) = -\ln(1-z)$$

$\star$ Results $\star$

- **Thermodynamic limit**

$$\mathrm{A}, \langle N \rangle \to \infty \qquad\qquad \langle n \rangle_{\mathrm{2D}} \equiv \frac{\langle N \rangle}{\mathrm{A}} \quad \text{fixed}$$

- **The range of fugacity** $\qquad D = 2 \qquad \alpha \neq 0$

$\odot \qquad 0 \leq z \leq z_0 < 1 \qquad \text{iff} \quad E_0 < 0$

$\odot \qquad 0 \leq z < 1 \qquad \text{iff} \quad E_0 \geq 0$

BEC occurs only in the presence of the bound state

only for the sub-family of the Self-Adjoint

Extensions of the symmetric Hamiltonian

$$-\infty < E_0 < 0$$

- **Critical quantities does not depend upon** α

$$\ln\left(1 - e^{\beta_c E_0}\right) = -\frac{h^2 \beta_c}{2\pi m} \langle n \rangle_{\mathrm{2D}}$$

Pure Contact Interaction in 2D

$$|\alpha| = 0 \qquad E_0 < 0$$

$$\langle n(\beta,\mu,E_0)\rangle_{\rm 2D} \quad = \lambda_T^{-2} g_1(z) + \frac{z}{\mathrm{A}(z_0 - z)}$$

$$-\frac{z}{\mathrm{A}} \int_0^\infty \frac{dE}{E}\; \frac{e^{-\beta E}}{1 - z\exp\{-\beta E\}} \frac{1}{\ln^2(-E/E_0) + \pi^2}$$

$$\Updownarrow$$

BEC occurs $\quad \forall E_0 < 0$

but for the free case $\quad E_0 = -\infty$

$\star$ **Critical quantities** $\star$

$$\ln\left(1 - e^{\beta_c E_0}\right) = -\frac{h^2 \beta_c}{2\pi m} \langle n\rangle_{\rm 2D}$$

Pure Contact Interaction in 3D

$|\alpha| = \frac{1}{2} \qquad E_0 \in \mathbf{R}$

$$\langle n(\beta,\mu,E_0)\rangle_{3\mathrm{D}} \; = \lambda_T^{-3} g_{3/2}(z) + \frac{z\vartheta(-E_0)}{\mathrm{V}(z_0 - z)} + \frac{z\vartheta(E_0)}{\mathrm{V}(1-z)}$$

$$-\frac{1}{8}\frac{z}{\mathrm{V}(1-z)} + \frac{z}{\mathrm{V}}\int_0^\infty \frac{dE}{2\pi E}\frac{\mathrm{sgn}(E_0)\sqrt{E|E_0|}e^{-\beta E}}{(E+|E_0|)(1-z\exp\{-\beta E\})}$$

$$\Updownarrow$$

BEC always exists in 3D

$\star$ **Critical quantities** $\star$

$$\lambda_{T_c}^3 \langle n\rangle_{3\mathrm{D}} = \zeta(3/2) \qquad\qquad E_0 \geq 0$$

$$\lambda_{T_c}^3 \langle n\rangle_{3\mathrm{D}} = g_{3/2}\left(e^{\beta_c E_0}\right) \qquad\qquad E_0 \leq 0$$

$$g_s(z) \equiv \frac{z}{\Gamma(s)}\int_0^\infty dx \; \frac{x^{s-1}\exp\{-\beta x\}}{1 - z\exp\{-\beta x\}}$$

3. UNIFORM FIELD

- **1-particle Hamiltonian in D spatial dimensions**

$$H_0^{(D)}(g) = \frac{\mathbf{p}^2}{2m} + mgx \qquad\qquad g \geq 0$$

- Notations

$$\mathbf{x} = (x_1, \ldots, x_D) \equiv (\mathbf{r}, x) \qquad \mathbf{p} = (p_1, \ldots, p_D) \equiv (\mathbf{k}, p)$$

◦ Boson Gas in the half-space $x \geq 0$

◦ **Hamiltonian bounded from below $\Leftrightarrow$ Stability**

- **Neumann's Boundary Conditions on the Floor**

$$\partial_x \psi(\mathbf{r}, x)|_{x=0} = 0$$

main conclusions do not depend on

specific boundary conditions on the floor

Spectrum & Eigenfunctions

- **Regular 1-particle Eigenfunctions**

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{\exp\{(i/\hbar)\,\mathbf{k}\cdot\mathbf{r}\}}{(2\pi\hbar)^{(D-1)/2}} \sqrt{\frac{-\kappa}{a'_n}}\,\frac{\mathrm{Ai}(\kappa x + a'_n)}{\mathrm{Ai}(a'_n)}$$

∘ $\mathrm{Ai}(x)$ **is the Airy's function**

$a'_n < 0$ **are the zeros of** $\mathrm{Ai}'(x)$

- **1-particle Spectrum**

$$E_{n,\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - E_g a'_n \qquad n \in \mathbf{N} \quad \mathbf{k} \in \mathbf{R}^{D-1}$$

$$\kappa \equiv \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \qquad E_g \equiv \frac{mg}{\kappa} = \frac{\hbar^2\kappa^2}{2m}$$

- **The spectrum is purely continuous if** $D \neq 1$

⋆ Translation Invariance in Transverse Space ⋆

average Number of particles

per **unit transverse Volume** $\mathrm{V}_\perp$

- **Thermodynamic limit**

$$\mathrm{V}_\perp, \langle N\rangle \to \infty \qquad\qquad \langle n\rangle_{\mathrm{D}} \equiv \frac{\langle N\rangle}{\mathrm{V}_\perp} \quad \text{fixed}$$

- **The range of chemical potential** $-\infty < \mu \leq -E_g a_1'$

- **The mean density of particles in the excited states**

$$\langle n_{\mathrm{ex}}(g,\beta,\mu)\rangle_{\mathrm{D}} \;=$$

$$\sum_{j=1}^{\infty} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}} \left\{ \exp\left[\beta \left(\frac{\mathbf{k}^2}{2m} - E_g a_j' - \mu \right)\right] - 1 \right\}^{-1}$$

$$= \; \lambda_T^{1-D} \sum_{j=1}^{\infty} g_{(D-1)/2} \left(\exp\left\{ \beta E_g a_j' + \beta\mu \right\} \right)$$

$$D > 1 \qquad\qquad \mu < -E_g a_1'$$

Absence of BEC for $D = 2, 3$

Look at the FIRST term in the sum

$$\lim_{\mu \uparrow -E_g a_1'} \; g_{(D-1)/2} \left(\exp \{ \beta E_g a_1' + \beta \mu \} \right) \;=\; \infty$$

iff $\qquad D = 2, 3$

$$\Updownarrow$$

$$\lim_{\mu \uparrow -E_g a_1'} \; \langle n_{\text{ex}}(g, \beta, \mu) \rangle_{\text{2D}} \;=\; \infty$$

$$\lim_{\mu \uparrow -E_g a_1'} \; \langle n_{\text{ex}}(g, \beta, \mu) \rangle_{\text{3D}} \;=\; \infty$$

KREIN'S FORMULA

- **Disorder** potential in D spatial dimensions

$$H = H_0 + V_{\rm dis}(\mathbf{r}) \qquad\qquad H_0 \equiv \frac{\mathbf{p}^2}{2m} + mgx$$

- Formal expansion of the Resolvent $\qquad w \in \mathbf{C}$

$$R(w) = [H - w]^{-1} = R_0(w) - \sum_{n=1}^{\infty} [R_0(w) V_{\rm dis}]^n R_0(w)$$

- Green's function of the *impurity free* Resolvent $D > 1$

$$G_0^{(D)}(w, g; \mathbf{x}, \mathbf{x}') \equiv \langle \mathbf{x} \,| R_0(w) |\, \mathbf{x}' \rangle = -\frac{\pi\kappa}{E_g} \int \frac{d^{D-1}k}{(2\pi\hbar)^{D-1}}$$

$$\exp\left\{ \frac{i}{\hbar} \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') \right\} \frac{u[\xi(x_<)]\, v[\xi(x_>)]}{\mathrm{Ai}'[\xi(0)]}$$

$$\Im\mathrm{m} z > 0 \qquad \mathbf{x} \neq \mathbf{x}' \qquad x_<(x_>) = \min(\max)\{x, x'\}$$

$$\xi \equiv \kappa x + \xi_0 \qquad \xi_0 \equiv {E_g}^{-1} \left(\frac{\mathbf{k}^2}{2m} - w \right)$$

$$u(\xi) \equiv \mathrm{Bi}'(\xi_0)\, \mathrm{Ai}(\xi) - \mathrm{Ai}'(\xi_0)\, \mathrm{Bi}(\xi) \qquad v(\xi) \equiv \mathrm{Ai}(\xi)$$

- **Point-like impurity** $\quad V_{\rm dis}(\mathbf{r}) = \lambda_D \delta^{(D)}(\mathbf{r})$

$$G^{(D)}(w, g, \lambda_D; \mathbf{x}, \mathbf{x}') =$$

$$G_0^{(D)}(w, g; \mathbf{x}, \mathbf{x}') - \frac{G_0^{(D)}(w, g; \mathbf{x}, \mathbf{0}) G_0^{(D)}(w, g; \mathbf{0}, \mathbf{x}')}{(1/\lambda_D) + G_0^{(D)}(w, g; \mathbf{0}, \mathbf{0})}$$

$$\Im \text{m} z > 0 \qquad \mathbf{x} \neq \mathbf{x}'$$

- As it stands the **denominator is meaningless** owing to **UV divergences at coincident points** of the *impurity free* Resolvent $\Rightarrow$ **Renormalization / Subtraction**

- *Impurity free* Green's function at $\mathbf{x} = \mathbf{x}' = 0$

$$G_0^{(D)}(\zeta, g; \mathbf{0}, \mathbf{0}) = -C_D \int_0^\infty dy \; y^{(D-3)/2} \, \frac{\text{Ai}(y - \zeta)}{\text{Ai}'(y - \zeta)}$$

$$C_D \equiv \frac{\kappa^D \; (4\pi)^{(1-D)/2}}{E_g \, \Gamma[(D-1)/2]} \qquad \zeta \equiv \frac{w}{E_g}$$

$$1 < \Re \text{e} D < 2$$

- Asymptotic behavior for large x

$$\frac{\text{Ai}(x)}{\text{Ai}'(x)} \sim x^{-1/2} \left[1 + O(x^{-3/2}) \right]$$

- **Renormalized** Green's function at $\mathbf{x} = \mathbf{x}' = 0$

$$G^{(D)}_{0,R}(\zeta, g, \eta; \mathbf{0}, \mathbf{0}) =$$

$$- C_D \int_0^\infty dy\, y^{(D-3)/2} \left[\frac{\mathrm{Ai}(y-\zeta)}{\mathrm{Ai}'(y-\zeta)} + (y+\eta)^{-1/2} \right]$$

$\eta > 0 \quad$ arbitrary $\qquad\qquad \Re \mathrm{e} D > 1$

- **Renormalized running** coupling parameter

$$\left[\lambda^R_D(\eta)\right]^{-1} = \lim_{\Lambda \to \infty} \left[\frac{1}{\lambda_D} - I_D(\Lambda, \eta) \right]$$

$$I_D(\Lambda, \eta) \equiv -C_D \int_0^\Lambda dy\ y^{(D-3)/2} \, (y+\eta)^{-1/2}$$

$$\Updownarrow$$

Renormalized denominator

of the Krein's formula

$$\frac{1}{\lambda_D} + G^{(D)}_0(\zeta, g; \mathbf{0}, \mathbf{0}) \equiv \frac{1}{\lambda^R_D} + G^{(D)}_{0,R}(\zeta, g, \eta; \mathbf{0}, \mathbf{0})$$

has one simple zero $\zeta_0 \equiv (E_0/E_g)$

in the interval $-\infty < \zeta_0 < -a'_1$

Singularities of the Resolvent

$\star$ **cut** along the real positive energy axis $\Re e w > -a'_1 E_g$

continuous spectrum

$\star$ **simple pole** on the real energy axis $\Re e w < -a'_1 E_g$

$$w_0 = E_0 = \zeta_0 E_g \qquad \forall \lambda_D^R \in \mathbf{R}$$

one bound state always exists

$\star$ **Complex poles** of the Resolvent $\forall \lambda_D^R \in \mathbf{R}$

resonances i.e. metastable states

the bound state energy E_0 labels

the one-parameter family of Self-Adjoint Extensions

of the symmetric Hamiltonian operator $H_0^{(D)}(g)$

- **Thermodynamic limit**

$$\mathrm{V}_\perp, \langle N \rangle \to \infty \qquad \langle n \rangle_{\mathrm{D}} \equiv \frac{\langle N \rangle}{\mathrm{V}_\perp} \quad \text{fixed}$$

- **The range of chemical potential when $\lambda_D^R \neq \pm\infty$**

$$-\infty < \mu \leq E_0 < -E_g a_1'$$

- **The mean density of particles in the excited states**

$$\langle n_{\mathrm{ex}} \rangle_{\mathrm{D}} = \lambda_T^{1-D} \sum_{j=1}^{\infty} g_{(D-1)/2} \left(\exp\left\{ \beta E_g a_j' + \beta\mu \right\} \right) < \infty$$

$$D = 2, 3 \qquad \forall \mu \leq E_0 < -E_g a_1'$$

$$\Updownarrow$$

BEC always occurs in $D = 2, 3$ iff $\lambda_D^R \neq \pm\infty$

⋆ Critical quantities ⋆

$$\lambda_{T_{\mathrm{c}}}^{D-1} \langle n \rangle_D \approx 4\pi \left(\kappa \lambda_{T_{\mathrm{c}}} \right)^{-3} g_{(D+2)/2} \left(e^{\beta_c E_0} \right)$$

4. CONCLUSIONS

- **Role of Disorder and External Fieds in the remarkable Physical Effects for Ideal Systems**
- **Exactly Solvable Models allow to grasp the key Dynamical Mechanisms leading to those Effects**
- **Point-like Interaction in QM is related to the Self Adjoint Extensions of the Symmetric Hamiltonian**
- **Contact Interaction fully corresponds to the δ-like Interaction only in $D = 1$**
- **Contact Interaction always provide Bose-Einstein condensation but for the case $D = 2$ $\alpha \neq 0$ $E_0 > 0$**
- **Bose-Einstein condensation never occurs with the Uniform Field and without Disorder in $D = 2, 3$**
- **Bose-Einstein condensation always occurs with the Uniform Field and Disorder $\forall D = 1, 2, 3$**
- **Critical Quantities have been explicitly computed**