

EXACT SOLUTION OF THE 1-IMPURITY QUANTUM HALL PROBLEM

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0. INTRODUCTION

1. SOLVED MODELS

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0. INTRODUCTION

- EXPERIMENTAL set of the Integer Quantum Hall Effect

Klaus von Klitzing (1980)

- QUANTIZED PLATEAUX IN $R_H = \sigma_{xy}^{-1}$

- VANISHING σ_{xx} IN QUANTUM HALL REGIME

- THEORETICAL "mesoscopic" explanation

- IDEAL 2D ELECTRON GAS

electron interactions are negligible

- GAUGE INVARIANCE (Laughlin 1984)

to explain R_H quantization

- DISORDER/IMPURITIES (Prange 1981)

to explain plateaux

- the 1-particle quantum mechanical problem to be solved

$$H = \frac{1}{2m} \left\{ \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{x}) \right\}^2 - \frac{e}{mc} \mathbf{s} \cdot \mathbf{B} - e\mathbf{x} \cdot \mathbf{E}_H + V_{\text{dis}}$$

- $\mathbf{s} = \frac{\hbar}{2} \vec{\sigma}$ is the electron spin operator
- $B > 0$ is the uniform magnetic field along the Ox_3 axis
- gauge potential in the symmetric gauge

$$A_j(x_1, x_2) = -\epsilon_{jl} x_l \frac{B}{2}, \quad j, l = 1, 2, \quad \epsilon_{12} = 1,$$

- the Hall field \mathbf{E}_H is uniform to a high degree of accuracy
- V_{dis} is the potential describing DISORDER/IMPURITIES

- Consider the PURE 2D IDEAL electron gas $V_{\text{dis}} = 0$

- ALL the electrons lie in the
HALL CONDUCTING LANDAU BANDS

- EACH STATE carries the Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{h}$$

- the TOTAL Hall conductance is

$$\Sigma_{xy} = -\frac{e^2}{h}\nu$$

$$\nu \equiv \frac{\langle \mathbf{n} \rangle}{\Gamma_L} \quad \text{Landau band filling}$$

$$\Gamma_L = \frac{eB}{hc} \quad \text{Landau degeneracy}$$

$$\langle \mathbf{n} \rangle = \text{average \# of electrons} \times \text{cm}^{-2}$$

THE P-J MECHANISM

Prange (1981) Joynt-Prange (1982)

- Consider the IMPURE 2D IDEAL electron gas $V_{\text{dis}} \neq 0$

○ there exist BOUND STATES which
DO NOT CARRY HALL CURRENT



○ the filling of Landau bands DECREASES

$$\nu \longmapsto \frac{\nu}{1 + \epsilon} \quad \epsilon > 0$$

○ the electron Hall conductivity just INCREASES

$$\sigma_{xy} = -\frac{e^2}{h}(1 + \epsilon)$$

- the net result of the P-J mechanism is therefore

○ THE HALL CONDUCTANCE IS AGAIN

$$\Sigma_{xy} = -\frac{e^2}{h}\nu$$

○ CHANGE IN DENSITY \Leftrightarrow FERMI ENERGY

$$2mE_F \simeq (h^3 \langle \mathbf{n} \rangle)^{2/3}$$

BETWEEN NEXT-TO-LYING LANDAU BANDS

\Updownarrow

○ NO CHANGE IN THE HALL CONDUCTANCE Σ_{xy}

\Updownarrow

○ **HALL PLATEAUX**

: QUESTION :

¿ IS IT POSSIBLE TO CHECK
THE P-J MECHANISM
IN SOME EXACTLY SOLVABLE MODEL ?

Prange (1981) SUGGESTED IT IS POSSIBLE
IF WE MODEL DISORDER IN TERMS OF
 V_{dis} WITH A **POINT-LIKE** SUPPORT

NOTATIONS

- holomorphic dimensionless coordinates

$$z = \frac{x_1 + ix_2}{\lambda_B} = x + iy$$

$$x_1 = \lambda_B \frac{z + \bar{z}}{2}, \quad x_2 = \lambda_B \frac{z - \bar{z}}{2i}$$

$$\partial_z = (\lambda_B/2) (\partial_1 - i\partial_2)$$

- $\lambda_B = \sqrt{\hbar c/eB}$ magnetic length
- $\varrho \equiv 2(E_H/B)\sqrt{mc^2/\hbar\omega}$ Hall parameter
- $\omega \equiv (eB/mc)$ cyclotron's angular frequency

1. SOLVED MODELS

- The PURE Landau model $V_{\text{dis}} = 0$ Landau (1930)

ENERGY creation-distruction operators

$$\delta_{\varrho} \equiv i\sqrt{2} \left\{ \partial_{\bar{z}} + \frac{z - \varrho}{4} \right\} = \bar{\delta}_{\varrho}^{\dagger}$$

$$\bar{\delta}_{\varrho} \equiv i\sqrt{2} \left\{ \partial_z - \frac{\bar{z} - \varrho}{4} \right\} = \delta_{\varrho}^{\dagger}$$

DEGENERACY creation-distruction operators

$$\theta_{\varrho} \equiv -i\sqrt{2} \left\{ \partial_z + \frac{\bar{z} - \varrho}{4} \right\} = \bar{\theta}_{\varrho}^{\dagger}$$

$$\bar{\theta}_{\varrho} \equiv -i\sqrt{2} \left\{ \partial_{\bar{z}} - \frac{z - \varrho}{4} \right\} = \theta_{\varrho}^{\dagger}$$

- OPERATOR ALGEBRA

$$[\delta_\varrho, \bar{\delta}_\varrho] = [\theta_\varrho, \bar{\theta}_\varrho] = 1, \quad [\delta_\varrho, \theta_\varrho] = [\delta_\varrho, \bar{\theta}_\varrho] = 0$$

- SPIN-UP (\uparrow) SCHRÖDINGER-PAULI HAMILTONIAN

$$H_\uparrow(E_H) \equiv \frac{\hbar^2}{2m\lambda_B^2} \mathbf{h}(\varrho)$$

- RESCALED HAMILTONIAN in units of energy ($\hbar^2/2m\lambda_B^2$)

$$\mathbf{h}(\varrho) = 2\bar{\delta}_\varrho\delta_\varrho + i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2$$

- DEGENERACY SPLITTING OPERATOR

$$\mathbf{T}(\varrho) \equiv i\frac{\varrho}{\sqrt{2}}(\bar{\theta}_\varrho - \theta_\varrho) - \frac{3}{4}\varrho^2$$

CONTINUOUS SPECTRUM & EIGENSTATES

$$\varepsilon_{n,p_{\perp}} = 2n - \varrho p_{\perp} - \frac{1}{4}\varrho^2$$

$$n + 1 \in \mathbf{N} \quad p_{\perp} = \frac{p_2 \lambda_B}{\hbar} \in \mathbf{R}$$

- IMPROPER NON-DEGENERATE eigenfunctions

$$\psi_{n,p_{\perp}}(x, y; \varrho) = u_n \left(x - \frac{1}{2}\varrho - p_{\perp} \right) \exp \left\{ iyp_{\perp} - i\frac{xy}{2} \right\} \sqrt{\frac{1}{2\pi}}$$

$\{u_k \mid k + 1 \in \mathbf{N}\}$ HERMITE's functions basis

- ORTHONORMALITY and COMPLETENESS

$$\langle \psi_{n,p_{\perp}}(\varrho) | \psi_{m,q_{\perp}}(\varrho) \rangle = \delta_{n,m} \delta(p_{\perp} - q_{\perp})$$

$$p_{\perp}, q_{\perp} \in \mathbf{R} \quad n + 1, m + 1 \in \mathbf{N}$$

- The Hall CURRENT OPERATOR is

$$J_H = \frac{ie\hbar}{m\lambda_B\sqrt{2}} \left(\delta_\varrho - \bar{\delta}_\varrho + \frac{i}{\sqrt{2}}\varrho \right)$$

EACH NON-DEGENERATE STATE $|\psi_{n,p_\perp}\rangle$
 CARRIES THE SAME HALL CURRENT

$$\langle \psi_{n,p_\perp} | J_H | \psi_{n,p_\perp} \rangle = \sigma_{xy} E_H$$



- the 1-ELECTRON STATE Hall conductivity is

$$\sigma_{xy} = -\frac{e^2}{h} \Gamma_L^{-1}$$

$$V_{\text{dis}} \neq 0 \quad 1\text{-IMPURITY MODEL} \quad \mathbf{E}_H = 0$$

1-IMPURITY described by
POINT-LIKE MAGNETIC VORTEX

Y. Aharonov, D. Bohm (1959)

A. Comtet, Y. Georgelin, S. Ouvry (1989)

J. Desbois, S. Ouvry, C. Texier (1997)

$$A_j(x_1, x_2) = -\epsilon_{jl} x_l \left[\frac{B}{2} - \frac{(\phi/2\pi)}{x_1^2 + x_2^2} \right]$$

◦ QUANTUM FLUX UNITY $\phi_0 \equiv (hc/e)$

◦ FLUX PARAMETER $\alpha \equiv (\phi/\phi_0)$

◦ INTEGER α is NOT OBSERVABLE $\Rightarrow -1 < \alpha \leq 0$

- SINGULAR creation-distruction operators $\gamma \equiv (\bar{z}z/2)$

$$\delta(\alpha) \equiv i\sqrt{2} \left\{ \partial_{\bar{z}} + \frac{z}{4} \left(1 - \frac{\alpha}{[\gamma]} \right) \right\} = \bar{\delta}^\dagger(\alpha)$$

$$\bar{\delta}(\alpha) \equiv i\sqrt{2} \left\{ \partial_z - \frac{\bar{z}}{4} \left(1 - \frac{\alpha}{[\gamma]} \right) \right\} = \delta^\dagger(\alpha)$$

$$\theta(\alpha) \equiv -i\sqrt{2} \left\{ \partial_z + \frac{\bar{z}}{4} \left(1 + \frac{\alpha}{[\gamma]} \right) \right\} = \bar{\theta}^\dagger(\alpha)$$

$$\bar{\theta}(\alpha) \equiv -i\sqrt{2} \left\{ \partial_{\bar{z}} - \frac{z}{4} \left(1 + \frac{\alpha}{[\gamma]} \right) \right\} = \theta^\dagger(\alpha)$$

(1/[γ])-SINGULARITY IS DEFINED
 ACCORDING TO THE
 THEORY OF DISTRIBUTIONS

$$\frac{1}{[\gamma]} = \frac{1}{4} \Delta \ln^2(\bar{z}z) + C\delta^{(2)}(\bar{z}, z)$$

C ARBITRARY CONSTANT

- DOMAIN of the SINGULAR operators

$$\mathcal{T}(\mathbf{R}^2) = \{f \in \mathcal{S}(\mathbf{R}^2) \mid f(0) = 0\}$$

dense in $L^2(\mathbf{R}^2)$

- SINGULAR OPERATORS ALGEBRA

$$[\delta(\alpha), \bar{\delta}(\alpha)] = [\theta(\alpha), \bar{\theta}(\alpha)] = 1$$

$$[\delta(\alpha), \theta(\alpha)] = [\delta(\alpha), \bar{\theta}(\alpha)] = 0$$

- ANGULAR MOMENTUM & hAMILTONIAN

$$\mathbf{L} = \hbar\{|\alpha|\mathbf{1} + \bar{\theta}(\alpha)\theta(\alpha) - \bar{\delta}(\alpha)\delta(\alpha)\}$$

$$\mathbf{h}(\alpha) = 2\bar{\delta}(\alpha)\delta(\alpha)$$

$$[\mathbf{h}(\alpha), \mathbf{L}] = 0$$

DISCRETE SPECTRUM & EIGENSTATES

- **INTEGER** valued ∞ -**DEGENERATE** LEVELS

$$\check{\epsilon}_n = 2n \quad L > 0$$

- **REGULAR** EXTENDED LANDAU BANDS

$$\Psi_{n < k}(z, \bar{z}) = (-1)^n \sqrt{\frac{n!}{2\pi\Gamma(k + \alpha + 1)}} \left(i \frac{z}{\sqrt{2}} \right)^{k-n}$$

$$\times \gamma^{-|\alpha|/2} \exp\{-\gamma/2\} L_n^{(k-n+\alpha)}(\gamma)$$

$$k \geq n + 1 \in \mathbf{N} \quad \ell = k - n > 0$$

$L_n^{(\beta)}$ Laguerre polynomials

- REAL valued $(\mathbf{n} + 1)$ -DEGENERATE LEVELS

$$\hat{\varepsilon}_n = 2(n - \alpha) \quad \mathbf{L} \leq 0$$

- REGULAR LOCALIZED EIGENSTATES

$$\Psi_{n \geq k}(z, \bar{z}) = (-1)^k \sqrt{\frac{k!}{2\pi\Gamma(n - \alpha + 1)}} \left(-i \frac{\bar{z}}{\sqrt{2}}\right)^{n-k}$$

$$\times \gamma^{|\alpha|/2} \exp\{-\gamma/2\} L_k^{(n-k-\alpha)}(\gamma)$$

$$n + 1 \in \mathbf{N} \quad 0 \leq k \leq n \quad \ell = k - n \leq 0$$

WITHOUT HALL FIELD $\mathbf{E}_H = 0$

THE WHOLE SET OF EIGENSTATES

IS ORGANIZED ACCORDING TO THE P-J PICTURE

: QUESTION :

ψ IS THIS **REGULAR** SOLUTION
THE **ONLY ONE** ALLOWED BY
GENERAL PRINCIPLES OF QM ?

THE ANSWER IS **NO**

2. SELF-ADJOINT EXTENSIONS

$$E_H = 0$$

- RADIAL HAMILTONIANS ARE ONLY **SYMMETRIC**

- THEOREM :

IN EACH SUBSPACE OF FIXED
INTEGER ANGULAR MOMENTUM $\ell \in \mathbf{Z}$
THE SYMMETRIC RADIAL HAMILTONIANS
ADMIT A 1-PARAMETER FAMILY OF
SELF-ADJOINT EXTENSIONS (SAE)

- DIFFERENT SAE HAVE DIFFERENT SPECTRA

- SAE **DOMAINS** CONTAIN L^2 -SINGULARITIES

CONTACT-INTERACTION

EXAMPLE : $\ell = 0$

$\alpha \neq 0$

- **SINGULAR COMPLETE O.N. SET $\in L^2(\mathbf{R}^2)$**

$$\Phi_{n,n}(z, \bar{z}) = (-1)^n \sqrt{\frac{n!}{2\pi\Gamma(n + \alpha + 1)}} \\ \times \gamma^{-|\alpha|/2} \exp\{-\gamma/2\} L_n^{(\alpha)}(\gamma) \quad n + 1 \in \mathbf{N}$$

- **RIESZ-FISHER EXPANSION ON THE REGULAR SET**

$$\Phi_{n,n}(z, \bar{z}) = \sum_{j=0}^{\infty} \check{c}_{n,0}^j \Psi_{j,j}(z, \bar{z}) \\ \check{c}_{n,0}^j \equiv \binom{j - \alpha}{n} \binom{n + \alpha}{j} \sqrt{\frac{j!n!}{\Gamma(j - \alpha + 1)\Gamma(n + \alpha + 1)}} \\ \Psi_{j,j}(z, \bar{z}) = (-1)^j \sqrt{\frac{j!}{2\pi\Gamma(j - \alpha + 1)}} \\ \times \gamma^{|\alpha|/2} \exp\{-\gamma/2\} L_j^{(-\alpha)}(\gamma) \quad j + 1 \in \mathbf{N}$$

SPECTRAL DECOMPOSITIONS $\ell = 0$

- REGULAR SAE :

$$h_0(\alpha) = \sum_{n=0}^{\infty} 2(n + |\alpha|) |\Psi_{n,n}\rangle \langle \Psi_{n,n}|$$

- SINGULAR SAE :

$$H_0(\alpha) = \sum_{n=0}^{\infty} 2n |\Phi_{n,n}\rangle \langle \Phi_{n,n}|$$

THEY CORRESPOND TO THE TWO
LINEARLY INDEPENDENT SOLUTIONS OF
THE 2nd ORDER DIFFERENTIAL OPERATOR
OF THE $\ell = 0$ RADIAL HAMILTONIAN
INTERPOLATING \Rightarrow 1-PARAMETER FAMILY

REMARKS

- THE SAME CONSTRUCTION $\forall \ell \in \mathbf{Z}$
- RESULT : ∞ -SET OF SAE
- SELECTED SAE ITEM :

$$\hat{\Delta}(\alpha) \equiv \sum_{n=0}^{\infty} 2(n + |\alpha|) \times$$

$$\left\{ \sum_{k=0}^n |\Psi_{n \geq k}\rangle \langle \Psi_{n \geq k}| + \sum_{k=n+1}^{\infty} |\Phi_{n < k}\rangle \langle \Phi_{n < k}| \right\}$$

- DISCRETE REAL-VALUED SPECTRUM
- ∞ -DEGENERATE ENERGY LEVELS
- REGULAR ($\ell \leq 0$) & SINGULAR ($\ell > 0$) EIGENSTATES

$$\hat{\Delta}(\alpha) \text{ SAE} \quad \hat{\varepsilon}_n = 2(n + |\alpha|)$$

7	\otimes	\otimes	\otimes	\otimes
6	\otimes	\otimes	\otimes	\otimes
5	\otimes	\otimes	\otimes	\otimes
4	\otimes	\otimes	\otimes	\otimes
3	\otimes	\otimes	\otimes	\odot
2	\otimes	\otimes	\odot	\odot
1	\otimes	\odot	\odot	\odot
0	\odot	\odot	\odot	\odot
$k^\uparrow / \hat{\varepsilon}_\rightarrow$	$2 \alpha $	$2 + 2 \alpha $	$4 + 2 \alpha $	$6 + 2 \alpha $

3. 1-IMPURITY EXACT SOLUTION

- the rescaled hAMILTONIAN

$$\frac{2m\lambda_B^2}{\hbar^2} H(\alpha, E_H) \equiv \mathbf{h}(\alpha, \varrho) + \mathbf{T}(\alpha, \varrho)$$

$$\mathbf{h}(\alpha, \varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$$

- DEGENERACY SPLITTING operator

$$\mathbf{T}(\alpha, \varrho) \equiv i\frac{\varrho}{\sqrt{2}} [\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)] - \frac{3}{4}\varrho^2$$

CREATION-DESTRUCTION ALGEBRA

$$\delta_\rho(\alpha) \equiv i\sqrt{2} \left\{ \partial_{\bar{z}} + \frac{z}{4} \left(1 - \frac{\alpha}{[\gamma]} \right) - \frac{\rho}{4} \right\}$$

$$\bar{\delta}_\rho(\alpha) \equiv i\sqrt{2} \left\{ \partial_z - \frac{\bar{z}}{4} \left(1 - \frac{\alpha}{[\gamma]} \right) + \frac{\rho}{4} \right\}$$

$$\theta_\rho(\alpha) \equiv -i\sqrt{2} \left\{ \partial_z + \frac{\bar{z}}{4} \left(1 + \frac{\alpha}{[\gamma]} \right) - \frac{\rho}{4} \right\}$$

$$\bar{\theta}_\rho(\alpha) \equiv -i\sqrt{2} \left\{ \partial_{\bar{z}} - \frac{z}{4} \left(1 + \frac{\alpha}{[\gamma]} \right) + \frac{\rho}{4} \right\}$$

$$[\delta_\rho(\alpha), \bar{\delta}_\rho(\alpha)] = [\theta_\rho(\alpha), \bar{\theta}_\rho(\alpha)] = 1$$

$$[\delta_\rho(\alpha), \theta_\rho(\alpha)] = [\delta_\rho(\alpha), \bar{\theta}_\rho(\alpha)] = 0$$

• COMMUTATION RELATIONS •

$$H(\alpha, \varrho) \equiv \mathbf{h}(\alpha, \varrho) + \mathbf{T}(\alpha, \varrho)$$

$$\mathbf{h}(\alpha, \varrho) = 2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$$

$$\mathbf{T}(\alpha, \varrho) \equiv i\frac{\varrho}{\sqrt{2}} [\bar{\theta}_\varrho(\alpha) - \theta_\varrho(\alpha)] - \frac{3}{4}\varrho^2$$

$$[H(\alpha, \varrho), \mathbf{h}(\alpha, \varrho)] = 0$$

$$[H(\alpha, \varrho), \mathbf{T}(\alpha, \varrho)] = 0$$

$$[H(\alpha, \varrho), \mathbf{L}] \neq 0$$

TO FIND a COMPLETE O.N. SET of

EIGENSTATES of $H(\alpha, \varrho)$

SEARCH for COMMON EIGENSTATES of

$$2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha) \quad \text{and} \quad \mathbf{T}(\alpha, \varrho)$$

- SAE of $2\bar{\delta}_\varrho(\alpha)\delta_\varrho(\alpha)$ have DISCRETE SPECTRUM
- $T(\alpha, \varrho)$ has CONTINUOUS SPECTRUM \Rightarrow
- ANY SAE of $H(\alpha, \varrho)$ has CONTINUOUS SPECTRUM

\Updownarrow

ALL THE EIGENSTATES MUST BE IMPROPER

\Updownarrow

PROPER BOUND STATES ARE FORBIDDEN

\Updownarrow

THERE IS A **UNIQUE**

ESSENTIALLY SELF-ADJOINT HAMILTONIAN

$$\hat{H}(\alpha, \varrho) = \hat{\Delta}(\alpha, \varrho) + T(\alpha, \varrho)$$

• THE SOLUTION •

$$\hat{H}(\alpha, \varrho) |n, p_{\perp}; \hat{\alpha}, \varrho\rangle = \left(2n - 2\alpha - \varrho p_{\perp} - \frac{1}{4}\varrho^2 \right) |n, p_{\perp}; \hat{\alpha}, \varrho\rangle$$

$$|n, p_{\perp}; \hat{\alpha}, \varrho\rangle \equiv \sum_{k=0}^{\infty} c_k^{(n)}(\tilde{p}) |n, k; \hat{\alpha}, \varrho\rangle$$

$$c_k^{(n)}(\tilde{p}) = (-i)^k u_k(\tilde{p}) \quad \tilde{p} \equiv p_{\perp} - \frac{1}{2}\varrho \quad \forall n+1 \in \mathbf{N}$$

$$|n, k; \hat{\alpha}, \varrho\rangle = \sqrt{\frac{\Gamma(1-\alpha)}{(k!)\Gamma(n+1-\alpha)}} [\bar{\delta}_{\varrho}(\alpha)]^n [\bar{\theta}_{\varrho}(\alpha)]^k |0, 0; \hat{\alpha}, \varrho\rangle$$

$$\langle z\bar{z}|0, 0; \hat{\alpha}, \varrho\rangle = \frac{\gamma^{|\alpha|/2} \exp\left\{-\frac{1}{4}(|z-\varrho|^2 - \varrho^2)\right\}}{\sqrt{2\pi\Gamma(1-\alpha)} {}_1F_1(1-\alpha, 1; \varrho^2/2)}$$

$$\langle n, p_{\perp}; \hat{\alpha}, \varrho|J_H|n, p_{\perp}; \hat{\alpha}, \varrho\rangle = -\frac{e^2}{h\Gamma_L} E_H$$

4. CONCLUSION

- MODEL OF DISORDER : POINT-LIKE VORTICES
- 1-IMPURITY HALL PROBLEM EXACTLY SOLVABLE
- WITHOUT HALL FIELD $\mathbf{E}_H = 0 \Rightarrow \infty \#$ of SAE
- WHEN $\mathbf{E}_H \neq 0$ THERE IS A **UNIQUE SAE**
- THE P-J PICTURE IS **NOT REALIZED**
- THE HALL CONDUCTANCE IS THE **VERY SAME**
- Σ_{xy} TOPOLOGICAL INVARIANT / C.F.T.