First Semester Course

Introduction to relativistic quantum field theory
(a primer for a basic education)

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0.1 Prologo

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Introduction : Some Notations

Here I like to say a few words with respect to the notation adopted. The components of all the four vectors have been chosen to be real. The metric is defined by means of the Minkowski's tensor

$$g_{\mu\nu} = \begin{cases} 0 & \text{for} \quad \mu \neq \nu \\ 1 & \text{for} \quad \mu = \nu = 0 \\ -1 & \text{for} \quad \mu = \nu = 1,2,3 \end{cases} \qquad \mu, \nu = 0,1,2,3$$

i.e. the invariant product of two four vectors a and b with components a^0, a^1, a^2, a^3 and b^0, b^1, b^2, b^3 is defined in the following manner

$$a \cdot b \equiv g_{\mu\nu}\, a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} = a^0 b^0 - a^k b^k$$

Summation over repeated indices is understood unless differently stated – Einstein's convention – bold type $\mathbf{a}, \mathbf{b}$ is used to denote ordinary space three–vectors. Indices representing all four components 0,1,2,3 are usually denoted by greek letters, while indices representing the three space components 1,2,3 are denoted by latin letters.

The upper indices are, as usual, contravariant while the lower indices are covariant. Raising and lowering indices are accomplished with the aid of the metric Minkowski's tensor , *e.g.*

$$a_\mu = g_{\mu\nu}\, a^\nu \qquad a^\mu = g^{\,\mu\nu}\, a_\nu \qquad g_{\mu\nu} = g^{\,\mu\nu}$$

Throughout the notes the natural system of units is used

$$c = \hbar = 1$$

for the speed of light and the reduced Planck's constant as well as the Heaviside–Lorentz C. G. S. system of electromagnetic units. In this system of units energy, momentum and mass have the dimensions of a reciprocal length or a wave number, while the time x^0 has the dimensions of a length. The Coulomb's potential for a point charge q is

$$\varphi(\mathbf{x}) = \frac{q}{4\pi\,|\,\mathbf{x}\,|} = \left(\frac{q}{e^2}\right) \frac{\alpha}{r}$$

and the fine structure constant is

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} = 7.297\,352\,568(24) \times 10^{-3} \approx \frac{1}{137}$$

The symbol $-e$ ($e > 0$) stands for the negative electron charge. We generally work with the four dimensional Minkowski's form of Maxwell's equation :

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \qquad \partial_\mu F^{\mu\nu} = J^\nu$$

where

$$A^\mu = (\varphi, \mathbf{A}) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

in which

$$\mathbf{E} = (E^1, E^2, E^3)\,, \qquad E^k = F^{k0} = F_{0k}\,, \quad (\,k = 1, 2, 3\,)$$

$$\mathbf{B} = (F_{23}, F_{31}, F_{12})$$

are the electric and magnetic fields respectively while

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}\,, \,\boldsymbol{\nabla}\right) = (\partial_t\,,\, \boldsymbol{\nabla}\,)$$

We will often work with the relativistic generalizations of the Schrödinger wave functions of 1-particle quantum mechanical states. We represent the energy momentum operators acting on such wave functions following the convention:

$$P^\mu = (P_0\,, \mathbf{P}) = \mathrm{i}\partial^\mu = (\mathrm{i}\partial_t\,, -\mathrm{i}\boldsymbol{\nabla})$$

In so doing the plane wave $\exp\{-\mathrm{i}\,p\cdot x\}$ has momentum $p^\mu = (\,p_0\,, \mathbf{p})$ and the Lorentz covariant coupling between a charged particle of charge q and the electromagnetic field is provided by the so called *minimal substitution*

$$P_\mu - q\,A_\mu(x) = \begin{cases} \mathrm{i}\partial_t - q\,\varphi(x) & \text{for} \quad \mu = 0 \\ \mathrm{i}\boldsymbol{\nabla} - q\,\mathbf{A}(x) & \text{for} \quad \mu = 1, 2, 3 \end{cases}$$

The Pauli spin matrices are the three hermitean 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy

$$\sigma_i\,\sigma_j = \delta_{ij} \;+\; \mathrm{i}\,\varepsilon_{ijk}\,\sigma_k$$

hence

$$\frac{1}{2}\,[\,\sigma_i\,, \sigma_j\,] \;=\; \mathrm{i}\,\varepsilon_{ijk}\,\sigma_k \qquad \frac{1}{2}\,\{\sigma_i\,, \sigma_j\} \;=\; \delta_{ij}$$

The Dirac matrices in the Weyl, or spinorial, or even chiral representation are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \qquad (\, k = 1, 2, 3\,)$$

with the hermiticity property

$$\gamma^{\mu\,\dagger} = \gamma^0\,\gamma^\mu\,\gamma^0 \qquad \{\gamma^\mu\,,\,\gamma^\nu\} = 2\,g^{\mu\nu}\,\mathbb{I}$$

Chapter 1

Basics in Group Theory

1.1 Groups and Group Representations

1.1.1 Definitions

A set G of elements $\mathrm{e}, f, g, h, \ldots \in G$ that satisfies all the four conditions listed below is called an *abstract group* or simply a *group*.

1. A law of composition, or multiplication, is defined for the set, such that the multiplication of each pair of elements f and g gives an element h of the set : this is written as

   $$fg = h$$

   Element h is called the product of the elements f and g, which are called the factors. In general the product of two factors depends upon the order of the factors, so that the elements fg and gf can be different.

2. The multiplication is associative : if f , g and h are any three elements, the product of the element f with gh must be equal to the product of the element fg with h

   $$f(gh) = (fg)h$$

3. The set G contains a unit element e called the identity giving the relation

   $$\mathrm{e}f = f\mathrm{e} = f \qquad \forall f \in G$$

4. For any element $f \in G$ there is an element $f^{-1} \in G$ called the inverse or reciprocal of f such that

   $$f^{-1}f = ff^{-1} = \mathrm{e} \qquad \forall f \in G$$

If the number of elements in G is finite, then the group is said to be finite, otherwise the group is called infinite. The number of elements in a finite group is named its order.

If the multiplication is commutative, *i.e.* , if for any pair of elements f and g we have $fg = gf$, then the group is said to be commutative or abelian.

Any subset of a group G , forming a group relative to the very same law of multiplication, is called a subgroup of G .

The one-to-one correspondence between the elements of two groups F and G

$$f \leftrightarrow g \qquad f \in F \quad g \in G$$

is said to be an *isomorphism* iff for any pair of relations

$$f_1 \leftrightarrow g_1 \quad f_2 \leftrightarrow g_2 \qquad f_1, f_2 \in F \quad g_1, g_2 \in G$$

then there follows the relation

$$f_1 f_2 \leftrightarrow g_1 g_2$$

Groups between the elements of which an isomorphism can be established are called *isomorphic groups*. As an example, consider the set of the $n-$th roots of unity in the complex plane

$$z_k = \exp\{\, 2\pi \mathrm{i}\, k/n \} \qquad (\, k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

which form the commutative group

$$\mathbb{Z}_n \equiv \left\{ z_k \in \mathbb{C} \,|\, z_k^n = z_0 = \mathrm{e}^{\,2\pi \mathrm{i} \jmath}\,,\ \jmath \in \mathbb{Z}\,,\ k = 0, 1, 2, \ldots, n-1\,,\ n \in \mathbb{N} \right\}$$

where the composition law is the multiplication

$$z_k \cdot z_h = z_{k+h} = z_{h+k} \qquad \forall\, k, h = 0, 1, 2, \ldots, n-1$$

the identity is $z_0 = \mathrm{e}^{\,2\pi \mathrm{i}\jmath}$ $(\,\jmath \in \mathbb{Z}\,)$ and the inverse $z_k^{-1} = \bar{z}_k$ is the complex conjugate. This finite group with n elements is isomorphic to the group of the counterclockwise rotations around the OZ axis $(\mathrm{mod}\, 2\pi)$, for instance, through the n angles

$$\varphi_k = \frac{2\pi k}{n} \quad (\, k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

in such a manner that

$$R_k = \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix} \qquad (\, k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

The isomorphism of these groups follow from the correspondence

$$\varphi_k \;\leftrightarrow\; z_k = \mathrm{e}^{\mathrm{i}\varphi_k} \qquad\qquad (\,k = 0, 1, 2, \ldots, n-1, \quad n \in \mathbb{N}\,)$$

Homomorphism between two groups differs from isomorphism only by the absence of the requirement of one-to-one correspondence : thus isomorphism is a particular case of homomorphism. For example, the group S_3 of the permutations of three objects is homomorphic to $\mathbb{Z}_2 = \{1, -1\}$, the law of composition being multiplication. The following relationships establish the homomorphism of the two groups : namely,

$$\left\{\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}\right\} \rightarrow 1 \qquad \left\{\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}\right\} \rightarrow 1 \qquad \left\{\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}\right\} \rightarrow 1$$

$$\left\{\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix}\right\} \rightarrow -1 \qquad \left\{\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}\right\} \rightarrow -1 \qquad \left\{\begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix}\right\} \rightarrow -1$$

The results of one branch of the theory of groups, namely the theory of group representations, are used in the overwhelming majority of important cases in which group theory is applied to Physics. The theory of the group representations studies the homomorphic mappings of an arbitrary abstract group on all possible groups of linear operators.

- We shall say that a representation T of a group G is given in a certain linear space L , iff to each element $g \in G$ there is a corresponding linear operator $T(g)$ acting in the space L , such that to each product of the elements of the group there is a corresponding product of the linear operators, *i.e.*

  $$T(g_1)T(g_2) = T(g_1 g_2) \qquad \forall g_1, g_2 \in G$$

  $$T(\mathrm{e}) = \mathbf{I} \qquad T(g^{-1}) = T^{-1}(g)$$

  where $\mathbf{I}$ denotes the identity operator on L . The dimensionality of the space L is said to be the dimensionality of the representation. A group can have representations both of a finite and of an infinite number of dimensions. By the very definition, the set of all linear operators $T(g) : L \rightarrow L \quad (\,g \in G\,)$ is closed under the multiplication or composition law. Hence it will realize an algebra of linear operators over L that will be denoted by $\mathfrak{A}(L)$. If the mapping $T \;:\; G \;\rightarrow\; \mathfrak{A}(L)$ is an isomorphism, then the representation T is said *faithful*. In what follows, keeping in mind the utmost relevant applications in Physics, we shall always assume that the linear spaces upon which the representations act are equipped by an inner product [1] , in such a manner that the concepts of orthonormality, adjointness and unitarity are well defined in the conventional way.

One of the problems in the theory of representations is to classify all the possible representations of a given group. In the study of this problem two concepts play a fundamental role : the concept of equivalence of representations and the concept of reducibility of representations.

- Knowing any representation T of a group G in the space L one can easily set up any number of new representations of the group. For this purpose, let us select any *non-singular invertible linear operator* A, carrying vectors from L into a space L' with an equal number d of dimensions, and assign to each element $g \in G$ the linear operator

$$T_A(g) = A\,T(g)A^{-1} \qquad \forall g \in G$$

acting in the vector space L'. It can be readily verified that the map $g \mapsto T_A(g)$ is a representation of the group G that will be thereby said *equivalent* to the representation $T(g)$.

All representations equivalent to a given one are equivalent to each other. Hence, all the representations of a given group split into classes of mutually equivalent ones. Accordingly, the problem of classifying all representations of a group is reduced to the more limited one of finding all mutually non–equivalent representations.

- Consider a subspace L_1 of the linear vector space L. The subspace $L_1 \subseteq L$ will be said an *invariant subspace* with respect to some given linear operator A acting on L iff

$$A\ell_1 \in L_1 \qquad \forall \ell_1 \in L_1$$

Of course L itself and the empty set $\varnothing$ are trivial invariant subspaces.

- The representation T of the group G in the vector space L is said to be *reducible* iff there exists in L at least one nontrivial subspace L_1 invariant with respect to *all* operators $T(g)$, $g \in G$. Otherwise the representation is called *irreducible*. All one dimensional representations are evidently irreducible.

[1] The inner product between to vectors $a, b \in L$ is usually denoted by $(a\,,\,b)$ in the mathematical literature and by the Dirac notation $\langle\, a \,|\, b\,\rangle$ in quantum physics. Here I shall often employ both notations without loss of clarity.

- A representation T of the group G in the vector space L is said to be *unitary* iff all the linear operators $T(g)$, $g \in G$, are unitary operators

$$T^{\dagger}(g) = T^{-1}(g) = T(g^{-1}) \qquad \forall g \in G$$

1.1.2 Theorems

There are two important theorems concerning unitary representations.

- Theorem 1. Let T a unitary reducible representation of the group G in the vector space L and let $L_1 \subseteq L$ an invariant subspace. Then the subspace $L_2 = \complement L_1$, the orthogonal complement of L_1, is also invariant.

  `Proof.`

  Let $\ell_1 \in L_1$, $\ell_2 \in L_2$, then $T^{-1}(g)\ell_1 \in L_1$ and $(T^{-1}(g)\ell_1, \ell_2) = 0$. On the other hand the unitarity of the representation T actually implies $(\ell_1, T(g)\ell_2) = 0$. Hence, $T(g)\ell_2$ is orthogonal to ℓ_1 and consequently $T(g)\ell_2 \in L_2$, $\forall g \in G$, so the theorem is proved. $\square$

  Hence, if the vector space L transforms according to a unitary reducible representation, it decomposes into two mutually orthogonal invariant subspaces L_1 and L_2 such that $L = L_1 \oplus L_2$. Iterating this process we inevitably arrive at the irreducible representations.

- Theorem 2. Each reducible unitary representation $T(g)$ of a group G on a vector space L decomposes, univocally up to equivalence, into the direct sum of irreducible unitary representations $\boldsymbol{\tau}_a(g)$, $a = 1, 2, 3, \ldots$, acting on the invariant vector spaces $L_a \subseteq L$ in such away that

$$L = L_1 \oplus L_2 \oplus L_3 \oplus \ldots = \bigoplus_a L_a \qquad L_a \perp L_b \text{ for } a \neq b$$

$$\boldsymbol{\tau}_a(g)\,\ell_a \in L_a \quad \forall a = 1, 2, \ldots \qquad \forall g \in G$$

  Conversely, each reducible unitary representation $T(g)$ of a group G can be always composed from the irreducible unitary representations $\boldsymbol{\tau}_a(g)$, $a = 1, 2, 3, \ldots$, of the group.

  The significance of this theorem lies in the fact that it reduces the problem of classifying all the unitary representations of a group G, up to equivalent representations, to that of finding all its irreducible unitary representations.

As an example of the decomposition of the unitary representations of the rotation group, we recall the decomposition of the orbital angular momentum which is well known from quantum mechanics. The latter is characterized by an integer $\ell = 0, 1, 2, \ldots$ and consist of $(2\ell+1) \times (2\ell+1)$ square matrices acting on quantum states of the system with given eigenvalues $\lambda_\ell = \hbar^2 \ell(\ell+1)$ of the orbital angular momentum operator $\mathbf{L}^2 = [\,\mathbf{r} \times (-i\hbar\nabla)\,]^2$. The latter are labelled by the possible values of the projections of the orbital angular momentum along a certain axis, *e.g.* $L_z = -\hbar\ell, -\hbar(\ell-1), \ldots, \hbar(\ell-1), \hbar\ell$, in such a manner that we have the spectral decomposition

$$\mathbf{L}^2 = \sum_{\ell=0}^{\infty} \hbar^2 \ell(\ell+1)\, \widehat{P}_\ell \qquad\qquad \widehat{P}_\ell = \sum_{m=-\ell}^{\ell} |\,\ell\, m\rangle\langle \ell\, m\,|$$

with

$$\langle\, \ell\, m\, |\, \ell'\, m'\,\rangle = \delta_{\ell\ell'}\,\delta_{mm'} \qquad\qquad \mathrm{tr}\, \widehat{P}_\ell = 2\ell + 1$$

where $[\,\mathrm{tr}\,]$ denotes the trace (sum over the diagonal matrix elements) of the projectors $\widehat{P}_\ell$ over the finite dimensional spaces L_ℓ $(\ell = 0, 1, 2, \ldots)$ spanned by the basis $\{\,|\,\ell\, m\rangle\,|\, m = -\ell, -\ell+1, \ldots, \ell-1, \ell\,\}$ of the common eigenstates of $\mathbf{L}^2$ and L_z . Thus, for each rotation g , which corresponds to a 3×3 orthogonal square matrix such that $g^{-1} = g^{\top}$, there exists a $(2\ell+1) \times (2\ell+1)$ unitary matrix $\boldsymbol{\tau}_\ell(g)$ that specifies how the $(2\ell+1)$ quantum states transform among themselves as a result of the rotation $g \in G$ and which actually realize all the irreducible unitary finite dimensional representations of the rotation group in the three–dimensional space. The Hilbert space $\mathfrak{H}$ for a point-like spinless particle is thereby decomposed according to

$$\mathfrak{H} = \bigoplus_{\ell=0}^{\infty} L_\ell \qquad L_\ell \perp L_m \text{ for } \ell \neq m \qquad \dim(L_\ell) = 2\ell + 1$$

$$\boldsymbol{\tau}_\ell \;:\; L_\ell \to L_\ell \qquad \ell = 0, 1, 2, \ldots$$

1.1.3 Direct Products of Group Representations

It is very useful to introduce the concepts of characters and the definition of product of group representations.

- Let $T(g)$ any linear operator corresponding to a given representation of the group G . We define the *characters* $\chi(g)$ of the representation to be the sum of the diagonal matrix elements of $T(g)$

$$\chi(g) \equiv \mathrm{tr}\, T(g) \qquad\qquad \forall\, g \in G \tag{1.1}$$

In the case of infinite dimensional representations we have to suppose the linear operators $T(g)$ to be of the trace class.

Two equivalent representations have the same characters, as the trace operation does not depend upon the choice of the basis in the vector space.

- **Definition**. Consider two representations $T_1(g)$ and $T_2(g)$ of the group G acting on the vector (Hilbert) spaces L_1 and L_2 of dimensions n_1 and n_2 respectively. Let

$$\{e_{1j} \in L_1 \,|\, j = 1, 2, \ldots, n_1\} \qquad \{e_{2r} \in L_2 \,|\, r = 1, 2, \ldots, n_2\}$$

any two bases so that

$$e_{1j\,2r} \equiv \{e_{1j} \otimes e_{2r} \,|\, j = 1, 2, \ldots, n_1\,,\ r = 1, 2, \ldots, n_2\}$$

is a basis in the tensor product $L_1 \otimes L_2$ of the two vector spaces with $\dim(L_1 \otimes L_2) = n_1\, n_2$. Then the matrix elements of the linear operators $T_1(g)$ and $T_2(g)$ with respect to the above bases will be denoted by

$$[\,T_1(g)\,]_{jk} \equiv \langle\, e_{1j}, T_1(g)\, e_{1k}\,\rangle \qquad [\,T_2(g)\,]_{rs} \equiv \langle\, e_{2r}, T_2(g)\, e_{2s}\,\rangle$$

The direct product

$$T(g) \equiv T_1(g) \times T_2(g) \qquad \forall\, g \in G$$

of the two representations is a representation of dimension $n = n_1\, n_2$ the linear operators of which, acting upon the tensor product vector space $L_1 \otimes L_2$, have the matrix elements, with respect to the basis $e_{1j\,2r}$, which are defined by

$$\begin{aligned} (\, e_{1j\,2r}\,,\, T(g)\, e_{1k\,2s}\,) &\equiv [\,T_1(g)\,]_{jk}\,[\,T_2(g)\,]_{rs} \\ &= (\, e_{1j}\,,\, T_1(g)\, e_{1k}\,)\ (\, e_{2r}\,,\, T_2(g)\, e_{2s}\,) \end{aligned} \tag{1.2}$$

where $j, k = 1, 2, \ldots, n_1$ and $r, s = 1, 2, \ldots, n_2$. From the definition (1.1) it is clear that we have

$$\chi(g) = \chi_1(g)\, \chi_2(g) \qquad \forall\, g \in G$$

because

$$\begin{aligned} \chi(g) &\equiv \sum_{j=1}^{n_1} \sum_{r=1}^{n_2} (\, e_{1j\,2r}\,,\, T(g)\, e_{1j\,2r}\,) \\ &= \sum_{j=1}^{n_1} (\, e_{1j}\,,\, T_1(g)\, e_{1j}\,) \sum_{r=1}^{n_2} (\, e_{2r}\,,\, T_2(g)\, e_{2r}\,) \\ &= \chi_1(g)\, \chi_2(g) \qquad \forall\, g \in G \end{aligned} \tag{1.3}$$

1.2 Continuous Groups and Lie Groups

1.2.1 The Continuous Groups

A group G is called *continuous* if the set of its elements forms a topological space. This means that each element $g \in G$ is put in correspondence with an infinite number of subsets $U_g \subset G$, called the *neighbourhoods* of any $g \in G$. This correspondence has to satisfy certain conditions that guarantee the full compatibility between the topological space structure and the group associative composition law - see the excellent monographies [5] for details. To illustrate the concept of neighbourhood we consider the group of the rotations around a fixed axis, which is an abelian continuous group. Let $g = R(\varphi)$, $0 \leq \varphi \leq 2\pi$, a rotation through an angle φ around *e.g.* the OZ axis. By choosing arbitrarily a positive number $\varepsilon > 0$, we consider the set $U_g(\varepsilon)$ consisting of all the rotations $g' = R(\varphi')$ satisfying the inequality $|\varphi - \varphi'| < \varepsilon$. Every such set $U_g(\varepsilon)$ is a neighbourhood of the rotation $g = R(\varphi)$ and giving ε all its possible real positive values we obtain the infinite manifold of the neighbourhoods of the rotation $g = R(\varphi)$.

The real functions $f : G \to \mathbb{R}$ over the group G is said to be *continuous for the element* $g_0 \in G$ if, for every positive number $\delta > 0$, there exists such a neighborhood U_0 of g_0 that $\forall g \in U_0$ the following inequality is satisfied

$$| f(g) - f(g_0) | < \delta \qquad \forall g \in U_0 \qquad g_0 \in U_0 \subset G$$

A continuous group G is called *compact* if and only if each real function $f(g)$, continuous for all the elements $g \in G$ of the group, is bounded. For example, the group of the rotations around a fixed axis is compact, the rotation group in the three dimensional space is also a compact group. On the other hand, the continuous abelian group $\mathbb{R}$ of all the real numbers is not compact, since there exist continuous although not bounded functions, *e.g.* $f(x) = x$, $x \in \mathbb{R}$. The Lorentz group is not compact.

A continuous group G is called *locally compact* if and only if each real function $f(g)$, continuous for all the elements $g \in G$ of the group, is bounded in every neighborhood $U \subset G$ of the element $g \in G$. According to this definition, the group of all the real numbers is a locally compact group and the Lorentz group is also a locally compact group.

- Theorem : if a group G is locally compact, it always admits irreducible unitary representations in infinite dimensional Hilbert spaces.

In accordance with this important theorem, proved by Gel'fand and Raikov, Gel'fand and Naïmark found all the unitary irreducible representations of the Lorentz group and of certain other locally compact groups.

In general, if we consider all possible continuous functions defined over a continuous group G , we may find among them some multi-valued functions. These continuous multi-valued functions can not be made single-valued *by brute force* without violating continuity, that is, by rejecting the superflous values for each element $g \in G$. As an example we have the function

$$f(\varphi) = \mathrm{e}^{\frac{1}{2}\,\mathrm{i}\varphi}$$

over the rotation group around a fixed axis. Since each rotation $g = R(\varphi)$ through an angle φ can also be considered as a rotation through an angle $\varphi + 2\pi$, this function must have two values for a rotation through the same angles : namely,

$$f_{+}(\varphi) = \mathrm{e}^{\frac{1}{2}\,\mathrm{i}\varphi} \qquad f_{-}(\varphi) = \mathrm{e}^{\frac{1}{2}\,\mathrm{i}\varphi + \mathrm{i}\pi} = -\,\mathrm{e}^{\frac{1}{2}\,\mathrm{i}\varphi} \;=\; -\,f_{+}(\varphi)$$

Had we rejected the second of these two values, then the function $f(\varphi)$ would become discontinuous at the point $\varphi = 0 = 2\pi$.

The continuous groups which admit continuous multi-valued functions are called *multiply connected*. As the above example shows, it turns out that the rotation group around a fixed axis is multiply connected. Also the rotation group in the three dimensional space is multiply-connected, as I will show below in some detail. The presence of multi-valued continuous functions in certain continuous groups leads us to expect that some of the continuous representations of these groups will be multi-valued. On the one hand, these multi-valued representations can not be ignored just because of its importance in many physical applications. On the other hand, it is not always possible, in general, to apply to those ones all the theorems valid for single-valued continuous representations.

To overcome this difficulty, we use the fact that every multiply connected group G is an homomorphic image of a certain simply connected group $\widetilde{G}$. It turn out that the simply connected group $\widetilde{G}$ can always be chosen in such a manner that none of its simply connected subgroups would have the group G as its homomorphic image. When the simply connected group $\widetilde{G}$ is selected in this way, it is called the *universal covering group* or the *universal enveloping group* of the multiply connected group G – see [5] for the proof. Consider once again, as an example, the multiply connected abelian group of the rotations around a fixed axis. The universal covering group for it is the simply connected commutative group $\mathbb{R}$ of all the real numbers. The homomorphism is provided by the relationship [2]

$$x \;\rightarrow\; \varphi = x - 2\pi\,[\,x/2\pi\,] \qquad -\infty < x < \infty \qquad 0 \le \varphi \le 2\pi$$

[2] We recall that any real number x can always be uniquely decomposed into its integer $[\,x\,]$ and fractional $\{x\}$ parts respectively.

where $x = [\,x\,] + \{x\}$. It turns out that every continuous representation of the group G , including any multi-valued one, can always be considered as a single-valued continuous representation of the universal enveloping group $\widetilde{G}$. The representations of the group G obtained in this manner do exhaust all the continuous representations of the group G .

1.2.2 The Lie Groups

The *Lie groups* occupy an important place among continuous groups.

> Marius Sophus Lie
> Nordfjordeid (Norway) 17.12.1842 – Oslo 18.02.1899
> *Vorlesungen über continuierliche Gruppen* (1893)

Lie groups occupy this special place for two reasons : first, they represent a sufficiently wide class of groups, including the most important continuous groups encountered in geometry, mathematical analysis and physics ; second, every Lie group satisfies a whole series of strict requirements which makes it possible to apply to its study the methods of the theory of differential equations. We can define a Lie group as follows.

- Let G some continuous group. Consider any neighborhood V of the unit element of this group. We assume that by means of n real parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ we can define every element of the neighborhood V in such a way that :

  1. there is a continuous one-to-one correspondence between all the different elements $g \in V$ and all the different $n-$ples of the real parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$;
  2. suppose that g_1, g_2 and $g = g_1 g_2$ lie the neighborhood V and that

  $$g_1 = g_1(\alpha'_1, \alpha'_2\,, \ldots, \alpha'_n) \qquad g_2 = g_2(\alpha''_1, \alpha''_2\,, \ldots, \alpha''_n)$$
  $$g = g\,(\alpha_1, \alpha_2\,, \ldots, \alpha_n)$$

  where

  $$\alpha_a = \alpha_a(\alpha'_1, \alpha'_2, \ldots, \alpha'_n\,;\, \alpha''_1, \alpha''_2\,, \ldots, \alpha''_n) \qquad a = 1, 2, \ldots, n$$

  then the functions α_a $(a = 1, 2, \ldots, n)$ are *analytic functions* of the parameters $\alpha'_b\,,\ \alpha''_c$ $(\,b, c = 1, 2, \ldots, n)$ of the factors.

  Then the continuous group G is called a Lie group of dimensions n. We shall always choose the real parameters α_a $(a = 1, 2, \ldots, n)$ in such a way that their zero values correspond to the unit element.

1.2.3 An Example : the Rotations Group

- The proper rotation group $SO(3)$.

Any rotation in the three–dimensional space can always be described by an oriented unit vector $\hat{\mathbf{n}}$ with origin in the center of rotation and directed along the axis of rotation and by the angle of rotation α . Therefore we can denote rotations by $g(\hat{\mathbf{n}},\alpha)$. The angle is measured in the counter–clockwise sense with respect to the positive direction of $\hat{\mathbf{n}}$. By elementary geometry it can be shown that any *active* rotation $g(\hat{\mathbf{n}},\alpha)$ transforms the position vector $\mathbf{r}$ into the vector

$$\mathbf{r}' = \mathbf{r}\,\cos\alpha + \hat{\mathbf{n}}\,(\mathbf{r}\cdot\hat{\mathbf{n}})(1-\cos\alpha) + \hat{\mathbf{n}}\times\mathbf{r}\,\sin\alpha$$

Every rotation is defined by three parameters. If we introduce the vector $\boldsymbol{\alpha} \equiv \alpha\,\hat{\mathbf{n}}$ $(0 \le \alpha < 2\pi)$, then we can take the projections of the vector $\boldsymbol{\alpha}$ on the coordinate axes as the three numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ that parametrize the rotation : namely

$$\begin{array}{ccc} \alpha_1 = \alpha\sin\theta\cos\phi & \alpha_2 = \alpha\sin\theta\sin\phi & \alpha_3 = \alpha\cos\theta \\ 0 \le \alpha < 2\pi & 0 \le \theta \le \pi & 0 \le \phi \le 2\pi \end{array} \tag{1.4}$$

These angular parameters are called the *canonical coordinates* of the rotation group and are evidently restricted to lie inside the 2–sphere

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2$$

Thus, the rotation group is a three dimensional Lie group.

Any (abstract) rotation g can be realized by means of a linear operator $R(g)$ that transforms the position vector $\mathbf{r}$ into the new position vector $\mathbf{r}' = R(g)\,\mathbf{r}$. This corresponds to the *active* point of view, in which the reference frame is kept fixed while the position vectors are moved. Of course, one can equivalently consider the *passive* point of view, in which the position vectors are kept fixed while the reference frame is changed. As a result, the linear operators corresponding to the two points of view are inverse one of each other. Historically, the *eulerian angles* φ, θ and ψ were firstly employed as parameters for describing rotations. A rotation can be represented as a product of three orthogonal matrices: the rotation matrix $R_3(\varphi)$ about the OZ axis, the rotation matrix $R_1(\theta)$ about the OX' axis, which is called the *nodal line*, and the rotation matrix $R_3(\psi)$ about the OZ' axis, *i.e.*

$$R(g) = R(\varphi,\theta,\psi) = R_3(\psi)\,R_1(\theta)\,R_3(\varphi)$$

We have

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_3(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From the parametrization in terms of the eulerian angles it immediately follows that every rotation matrix is univocally identified by a tern

$$g \;\leftrightarrow\; R(\varphi, \theta, \psi) \qquad 0 \le \varphi < 2\pi \;;\; 0 \le \theta \le \pi \;;\; 0 \le \psi < 2\pi \tag{1.5}$$

and that

$$R\,(g^{-1}) = R^{-1}(g) = R_3(-\varphi)\, R_1(-\theta)\, R_3(-\psi) = R^{\top}(g)$$

with $\det\,[\,R(\varphi, \theta, \psi\,)\,] = 1$. It follows thereby that an isomorphism exists between the abstract rotation group and the group of the orthogonal 3×3 matrices with unit determinant : this matrix group is called the 3–dimensional special orthogonal group and is denoted by $SO(3)$.

The discrete transformation that carries every position vector $\mathbf{r}$ into the vector $-\,\mathbf{r}$ is called inversion or parity transform. Inversion I commutes with all rotations $Ig = gI$, $\forall g \in SO(3)$; moreover $I^2 = \mathrm{e}$, $\det I = -1$. If we add to the elements of the rotation group all possible products Ig , $g \in SO(3)$ we obtain a group, as it can be readily checked.

This matrix group is called the 3–dimensional *full orthogonal group* and is denoted by $O(3)$. The group $O(3)$ splits into two connected components : the proper rotation group $O(3)^+ = SO(3)$, which is the subgroup connected with the unit element, and the the component $O(3)^-$ connected with the inversion, wich is not a group. Evidently the matrices belonging to $O(3)^{\pm}$ have determinant equal to ± 1 respectively.

- The full orthogonal group $O(3)$ is the group of transformations that leave invariant the line element

$$\mathrm{d}\mathbf{r}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}x^i \mathrm{d}x^i \qquad i = 1, 2, 3$$

in the 3–dimensional space. In a quite analogous way one introduces, for odd N , the $N-$dimensional full orthogonal groups $O(N)$ and the proper orthogonal Lie groups $SO(N)$ of dimensions $n = \frac{1}{2}\,N(N-1)$.

1.2.4 The Infinitesimal Operators

- Infinitesimal operators : in the following we shall consider only those representations T of a Lie group G, the linear operators of which are *analytic funtions* of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ in such a way that

  $$T(g) = T(\alpha_1, \alpha_2, \ldots, \alpha_n) \equiv T(\alpha) : V \to \mathfrak{A}(V) \qquad (V \subset G)$$

  are analytic functions of their arguments.

  The derivative of the operator $T(g)$ with respect to the parameter α_a taken for $g = \mathrm{e}$, *i.e.* at the values $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, is called the *infinitesimal operator or generator* I_a of the representation $T(g)$ corresponding to the parameter α_a.

  Thus, any representation $T(g)$ has n generators, *i.e.* , the number of the infinitesimal operators is equal to the dimensionality of the Lie group.

- As an example let us consider the orthogonal matrices referred to the canonical coordinates (1.4) of the rotation group. Since the orthogonal matrices $R \in SO(3)$ are linear operators acting on the three–vectors $\mathbf{r} \in L = \mathbb{R}^3$, it follows that the orthogonal matrices do indeed realize a representation of the abstract rotation group with the same number of dimensions as the group itself.

  Thereby, the orthogonal matrices do realize the so–called *vector* or *adjoint* representation $\boldsymbol{\tau}_A(g)$ of the rotation group.

  According to the passive point of view, the rotation g with the canonical coordinates $(\alpha_1, 0, 0)$ corresponds to the rotation of the reference frame around the positive direction of the OX axis through an angle α_1. Then the operator $\boldsymbol{\tau}_A(\alpha_1, 0, 0)$ is represented by the matrix

  $$\boldsymbol{\tau}_A(\alpha_1, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$$

  Hence

  $$\frac{\partial}{\partial\alpha_1}\boldsymbol{\tau}_A(\alpha_1, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \\ 0 & -\cos\alpha_1 & -\sin\alpha_1 \end{pmatrix}$$

  and consequently

  $$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

In a quite analogous way we obtain

$$I_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The following commutation relations among the infinitesimal operators of the rotation group $SO(3)$ hold true : namely,

$$I_a I_b - I_b I_a = -\,\varepsilon_{abc}\, I_c \qquad (a,b,c = 1,2,3) \tag{1.6}$$

where ε_{abc} is the completely antisymmetric Levi–Civita symbol with the normalization $\varepsilon_{123} = 1$.

The crucial role played by the generators in the theory of the Lie groups is unraveled by the following three theorems :

- **Theorem I** . Let T_1 and T_2 any two representations of a Lie group G acting on the same vector space L and suppose that they have the same set of generators. Then T_1 and T_2 are the same representation.

- **Theorem II** . The infinitesimal operators I_a $(a = 1,2,\ldots,n)$ that correspond to any representation $T(g)$ of a Lie group G do satisfy the following commutation relations

$$I_a I_b - I_b I_a \equiv [\, I_a\, I_b\,] = C_{abc}\, I_c \qquad a,b,c = 1,2,\ldots,n \tag{1.7}$$

where the constant coefficients $C_{abc} = -\,C_{bac}$ do not depend upon the choice of the representation $T(g)$. The constant coefficients C_{abc} of a Lie algebra fulfill the Jacobi's identity

$$C_{abe}C_{ecd} + C_{bce}C_{ead} + C_{cae}C_{ebd} = 0 \tag{1.8}$$

because of the identity

$$[\,[\, I_a\, I_b\,]\, I_c\,] + [\,[\, I_b\, I_c\,]\, I_a\,] + [\,[\, I_c\, I_a\,]\, I_b\,] = 0$$

which can be readily checked by direct inspection.

- **Theorem III** . Suppose that a set of linear operators $A_1, A_2, \ldots, A_n$ acting on a certain vector space L is given and which fulfill the same commutation relations

$$[\, A_a\, A_b\,] = C_{abc}\, A_c \qquad a,b,c = 1,2,\ldots,n$$

as the infinitesimal operators of the group G. Then the operators $A_a (a = 1, 2, \ldots, n)$ are the generators of a certain representation $T(g)$ of the group G in the vector space L.

The details of the proofs of the above three fundamental theorems of the theory of the Lie groups can be found in the excellent monographies [5]. Theorems I, II and III are very important because they reduce the problem of finding all the representations of a Lie group G to that of classifying all the possible sets of linear operators which satisfy the commutation relations (1.7).

- Lie algebra : the infinitesimal operators I_a $(a = 1, 2, \ldots, n)$ do generate a linear space and the commutators define a product law within this linear space. Then we have an algebra which is called the *Lie algebra* $\mathcal{G}$ of the Lie group G with $\dim \mathcal{G} = \dim G = n$. The constant coefficients $C_{abc} = -\,C_{bac}$ are named the *structure constants* of the Lie algebra $\mathcal{G}$.

  Suppose that the representation $T(g)$ of the Lie group G acts on the linear space L and let A any invertible linear operator upon L. Then the linear operators $A\, I_a\, A^{-1} \equiv J_a$ $(a = 1, 2, \ldots, n)$ do realize an equivalent representation for $\mathcal{G}$, *i.e.* the infinitesimal operators J_a $(a = 1, 2, \ldots, n)$ correspond to a *new basis* in $\mathcal{G}$ because

$$
\begin{aligned}
[\, J_a\, J_b\,] &= [\, A\, I_a\, A^{-1}\, A\, I_b\, A^{-1}\,] = A\,[\, I_a\, I_b\,]\, A^{-1} \\
&= A\, C_{abc}\, I_c\, A^{-1} = C_{abc}\, A\, I_c\, A^{-1} \\
&= C_{abc}\, J_c \qquad\qquad a, b, c = 1, 2, \ldots, n
\end{aligned}
$$

  As a corollary, it turns out that, for any representation, the collections of infinitesimal operators

$$
\left\{ I_a(g) \equiv T(g)\, I_a\, T^{-1}(g) \,|\, a = 1, 2, \ldots, n \right\} \qquad \forall\, g \in G \tag{1.9}
$$

  do indeed realize different and equivalent choices of basis in the Lie algebra. We will say that the operators $T(g)$ of a given representation of the Lie group G generate all the *inner automorphisms* in the given representation of the Lie algebra $\mathcal{G}$.

- It is possible to regard the structure constants as the matrix elements of the n–dimensional representation of the generators : namely,

$$
\|\; I_a \;\|_{\,bc} = C_{acb} \qquad (a, b, c = 1, 2, \ldots, n) \tag{1.10}
$$

  in such a way that we can rewrite the Jacobi identity (1.8) as the matrix identity

$$
-\,||\, I_c \,||_{\,de}\, ||\, I_a \,||_{\,eb} + ||\, I_a \,||_{\,de}\, ||\, I_c \,||_{\,eb} - \,||\, I_c \,||_{\,ea}\, ||\, I_e \,||_{\,db} = 0 \tag{1.11}
$$

or after relabeling of the indices

$$|| [I_a \, I_b] ||_{de} = C_{abc} \, || \, I_c \, ||_{de}$$

Thus, for each Lie algebra $\mathcal{G}$ and Lie group G of dimensions n , there is a representation, called the *adjoint representation*, which has the very same dimensions n as the Lie group itself.

Evidently, an abelian Lie group has vanishing structure constants, so that its adjoint representation is trivial, *i.e.* it consists of solely the unit element.

As shown above the rotation group has three generators I_a $(a = 1, 2, 3)$ and structure constants C_{abc} equal to $-\,\varepsilon_{abc}$. Consequently, the adjoint representation of the rotation group is a three dimensional one with generators I_a given by

$$|| \, I_a \, ||_{bc} \;=\; \varepsilon_{abc} \qquad\qquad (a, b, c = 1, 2, 3)$$

In any neighborhood of the identity operator $T(\mathrm{e}) = T(0, 0, \ldots, 0) \equiv \mathbf{1}$ we can always find a set of parameters such that

$$T(\alpha) = \exp\, \{ I_a \, \alpha_a \} = \mathbf{1} + \sum_{a=1}^{n} I_a \, \alpha_a + O(\alpha^2\,) \tag{1.12}$$

Thus, in order to find the representations of the group G it is sufficient to classify the representations of its Lie algebra $\mathcal{G}$. More precisely, one can prove the following theorem – see [5] for the proof.

1.2.5 The Canonical Coordinates

- Exponential representation : let I_a $(a = 1, 2, \ldots, n)$ a given base of the Lie algebra $\mathcal{G}$ of a Lie group G . Then, for any neighborhood $U \subset G$ of the unit element $\mathrm{e} \in G$ there exists a set of *canonical coordinates* α_a $(a = 1, 2, \ldots, n)$ in U and a positive number $\delta > 0$ such that

  $$T(g) \equiv T(\alpha_1, \alpha_2\,, \ldots, \alpha_n) = \exp\, \{ I_a \, \alpha_a \} \qquad g \in U \subset G\,,\ |\, \alpha_a \,| < \delta$$

  where the exponential is understood by means of the so–called *Baker–Campbell–Hausdorff* formula

  $$T(\alpha)\, T(\beta) = T(\gamma) \qquad T(\alpha) = \exp\, \{ I_a \, \alpha_a \} \quad T(\beta) = \exp\, \{ I_b \, \beta_b \}$$

$$
\begin{aligned}
T(\gamma) \;\; &\equiv \;\; \exp\left\{ I_a\,\alpha_a + I_b\,\beta_b + \tfrac{1}{2}\,\alpha_a\beta_b\,[\,I_a\,I_b\,] \right. \\
&+ \;\; \left. \tfrac{1}{12}\,(\alpha_a\alpha_b\beta_c + \beta_a\beta_b\alpha_c)\,[\,I_a\,[\,I_b\,I_c\,]\,] + \cdots \right\}
\end{aligned}
$$

in which the dots stand for higher order, iterated commutators among generators. To elucidate these notions let me discuss two examples.

1. Consider the translations group along the real line

   $$x \;\rightarrow\; x' = x + a \qquad\qquad (\,\forall\, x \in \mathbb{R}\,, \quad a \in \mathbb{R}\,)$$

   The translations group on the real line is a one dimensional abelian Lie group. Let us find the representation of this group acting on the infinite dimensional functional space of all the real analytic functions $f\,:\,\mathbb{R} \rightarrow \mathbb{R}$ with $f \in C_\infty(\mathbb{R})$. To do this, we have to recall the Taylor-McLaurin formula

   $$f(x+a) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,f^{\,(k)}(x)$$

   Now, if we define

   $$T\,f(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} \qquad\qquad T^k\,f(x) \equiv \frac{\mathrm{d}^k f}{\mathrm{d}x^k} = f^{\,(k)}(x)$$

   then we can write

   $$
   \begin{aligned}
   f(x+a) \;\; &= \;\; \exp\{\,a\cdot T\,\}\,f(x) \\
   &= \;\; \left(\sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,T^k \right) f(x) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,a^k\,f^{\,(k)}(x)
   \end{aligned}
   $$

   in such a manner that we can actually identify the generator of the infinite dimensional representation of the translations group with the derivative operator acting upon the functional space of all the real analytic functions.

2. As a second example, consider the rotations group on the plane around a fixed point, which is an abelian and compact Lie group $O(2,\mathbb{R})$ *i.e.* the two dimensional real orthogonal group. A generic group element $g \in O(2)$ is provided by the 2×2 orthogonal matrix

   $$g(\varphi) = \left(\begin{array}{cc} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{array} \right) \qquad\qquad (\,0 \le \varphi \le 2\pi\,)$$

which corresponds to a passive planar rotation around the origin. If we introduce the generator

$$t \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $t^2 = -\,\mathbf{I}$, then we readily find

$$\begin{aligned} g(\varphi) &\equiv \exp\{\,t\,\varphi\,\} = \sum_{k=0}^{\infty} \frac{1}{k\,!}\,\varphi^k\,t^k \\ &= \sum_{k=0}^{\infty} (-1)^k \left[\frac{\varphi^{2k}}{(2k)\,!}\,\mathbf{I} \;+\; \frac{\varphi^{2k+1}}{(2k+1)\,!}\;t \right] \\ &= \mathbf{I}\,\cos\varphi + t\,\sin\varphi \qquad\qquad \forall\,\varphi \in [\,0\,,\,2\pi\,] \end{aligned}$$

The exponential representation of the Lie group elements implies some remarkable features.

- The generators of a unitary representation are *antihermitean*

$$T(g^{-1}) = T^{\,\dagger}(g) \quad \Leftrightarrow \quad I_a = -\,I_a^{\dagger} \quad (a = 1, 2, \ldots, n)$$

- All the structure constants of an abelian Lie group are equal to zero.

- For a compact Lie group G with $\dim G = n$ it is possible to show that all the sets of canonical coordinates span a bounded subset of $\mathbb{R}^n$. Moreover any element of any representation of a compact group can always be expressed in the exponential form. In a non-compact Lie group G with $\dim G = n$ the canonical coordinates run over an unbounded subset of $\mathbb{R}^n$ and, in general, not all the group elements can be always expressed in the exponential form.

- Any two Lie groups G_1 and G_2 with the same structure constants are *locally homeomorphic*, in the sense that it is always possible to find two neighborhoods of the unit elements $U_1 \subset G_1$ and $U_2 \subset G_2$ such that there is an analytic isomorphism $f\,:\,U_1 \leftrightarrow U_2$ between the elements of the two groups $\forall\,g_1 \in U_1\,,\ \forall\,g_2 \in U_2$. Of course, this does not imply that there is a one–to–one analytic map over the whole parameter space, *i.e.* the two groups are not necessarily *globally homeomorphic*. As an example, consider the group $\mathbb{R}$ of all the real numbers and the unitary group

$$U(1) \equiv \{z \in \mathbb{C}\,|\,z\,\bar{z} = 1\}$$

of the complex unimodular numbers. These two groups are abelian groups. In a neighborhood of the unit element we can readily set up an homeomorphic map, *e.g.* the exponential map

$$z = \exp\{\mathrm{i}\alpha\} \qquad \bar{z}z = 1 \qquad -\pi < \alpha \leq \pi$$

so that the angle α is the canonical coordinate. However, there is no such mapping in the global sense, because the unit circle in the complex plane is equivalent to a real line of which all the elements modulo 2π are considered as identical. In other words, we can associate any real number $x = 2k\pi + \alpha$ ($k \in \mathbb{Z}$, $-\pi < \alpha \leq \pi$) to some point of the unit circle in the complex plane with a given canonical coordinate α and a given integer winding number $k = [\,x/2\pi\,]$. If this is the case, the complex unit circle is not *simply connected*, because all the closed paths on a circle have a non–vanishing integer number of windings and can not be continuously deformed to a point, at variance with regard to the simply connected real line.

1.2.6 The Special Unitary Groups

- Example : the unitary group $SU(2)$. Consider the group of the 2×2 complex unitary matrices with unit determinant denoted as $SU(2)$, Special Unitary 2-dimensional matrices

$$g = \begin{pmatrix} \bar{v} & -\bar{u} \\ u & v \end{pmatrix} \qquad \bar{u}u + \bar{v}v = 1$$

which evidently realize a Lie group of dimensions $n = 3$. Notice that we can always set

$$u = -x_2 + \mathrm{i}x_1 \qquad v = x_4 + \mathrm{i}x_3 \qquad \sum_{i=1}^{4} x_i^2 = 1$$

whence it manifestly follows that the group $SU(2)$ is topologically homeomorphic to the three dimensional hypersphere S_3 of unit radius plunged in $\mathbb{R}^4$.

As all the $SU(2)$ matrices are unitary, the three generators are anti-hermitean 2×2 matrices, which can be written in terms of the Pauli matrices

$$\tau_a \equiv \tfrac{1}{2}\,\mathrm{i}\,\sigma_a \qquad (a = 1, 2, 3) \tag{1.13}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.14}$$

and therefore

$$[\,\tau_a\,\tau_b\,] = -\,\varepsilon_{abc}\,\tau_c \qquad (a,b,c = 1,2,3) \tag{1.15}$$

It follows that $SU(2)$ has the very same Lie algebra (1.6) of the rotation group. Using the canonical coordinates (1.4), in a neighborhood of the unit element we can write the $SU(2)$ elements in the exponential representation. From the very well known identities

$$\sigma_a\,\sigma_b + \sigma_b\,\sigma_a = 2\,\delta_{ab} \qquad (\sigma_a\,\alpha_a)^2 = |\,\boldsymbol{\alpha}\,|^{\,2} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

where the 2×2 identity matrix is understood, we obtain

$$\begin{aligned} g(\alpha) &= \exp\{\tau_a\,\alpha_a\} \equiv \sum_{k=0}^{\infty} \frac{1}{k!}\,(\tau_a\,\alpha_a)^k \\ &= \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k)!}\left(\frac{\mathrm{i}}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k} + \frac{1}{(2k+1)!}\left(\frac{\mathrm{i}}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k+1} \frac{\sigma_a\,\alpha_a}{|\,\boldsymbol{\alpha}\,|} \right\} \\ &= \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!}\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k} + \frac{\mathrm{i}\,(-1)^k}{(2k+1)!}\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right)^{2k+1} \frac{\sigma_a\,\alpha_a}{|\,\boldsymbol{\alpha}\,|} \right\} \\ &= \cos\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right) + \mathrm{i}\,\sigma_a\,\frac{\alpha_a}{|\,\boldsymbol{\alpha}\,|}\,\sin\left(\frac{1}{2}\,|\,\boldsymbol{\alpha}\,|\right) \\ &= \mathbf{1}\,x_4 + \mathrm{i}\sigma_a\,x_a \qquad (x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1) \end{aligned} \tag{1.16}$$

Notice that, by direct inspection,

$$g^{\dagger}(\alpha) = g(-\alpha) = g^{-1}(\alpha)$$

so that $g(\alpha)$ are unitary matrices. Furthermore

$$g(\alpha) = \begin{pmatrix} x_4 + \mathrm{i}x_3 & \mathrm{i}x_1 + x_2 \\ \mathrm{i}x_1 - x_2 & x_4 - \mathrm{i}x_3 \end{pmatrix}$$

hence $\det g(\alpha) = 1$. From the explicit formula (1.16) it follows that the whole set of special unitary 2×2 matrices is spanned if and only if the canonical coordinates $\boldsymbol{\alpha}$ are restricted to lie inside a sphere of radius 2π

$$|\,\boldsymbol{\alpha}\,|^{\,2} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2$$

it follows that the group $SU(2)$ realizes the lowest dimensional non-trivial, faithful, irreducible and unitary representation of the rotation group, which is thereby named its *fundamental representation* and denoted by $\boldsymbol{\tau}_F(g)$. Since the fundamental representation acts upon complex two–components column vectors, the Pauli spinors of non–relativistic quantum mechanics that describe particles of spin $\frac{1}{2}$, it is also named as the *spinorial representation* of the rotation group and further denoted by $\boldsymbol{\tau}_{\frac{1}{2}}(g) \equiv \boldsymbol{\tau}_F(g)$.

- A comparison between the fundamental $\boldsymbol{\tau}_F(g) = SU(2)$ and the adjoint $\boldsymbol{\tau}_A(g) = SO(3)$ representations of the rotation group is very instructive. Since these two representations of the rotation group share the same Lie algebra they are locally homeomorphic. It is possible to obtain all the finite elements of $SO(3)$ in the exponential representation. As a matter of fact, for the infinitesimal operators (1.6) of the adjoint representation we have (the identity matrix is understood)

$$(I_a\,\alpha_a)^2 = -\,|\,\boldsymbol{\alpha}\,|^{\,2} + \Xi(\alpha) \qquad ||\,\Xi(\alpha)\,||_{\,ac} \equiv \alpha_a\alpha_c$$

$$(I_a\,\alpha_a)^3 \;=\; -\,|\,\boldsymbol{\alpha}\,|^{\,2}\,I_a\,\alpha_a$$

and consequently

$$[\,\Xi(\alpha)\,]^2 = |\,\boldsymbol{\alpha}\,|^{\,2}\,\Xi(\alpha) \qquad\qquad \alpha_a\,I_a\,\Xi(\alpha) = 0 = \Xi(\alpha)\,I_a\,\alpha_a$$

$$(I_a\,\alpha_a)^{\,2k} = (-\,|\,\boldsymbol{\alpha}\,|^{\,2})^{\,k}\Big(\mathbf{1} - |\,\boldsymbol{\alpha}\,|^{-2}\,\Xi(\alpha)\Big)$$

$$(I_a\,\alpha_a)^{\,2k+1} = (-\,|\,\boldsymbol{\alpha}\,|^{\,2})^{\,k}\,I_a\,\alpha_a$$

in such a way that

$$\begin{aligned}
\boldsymbol{\tau}_A(\alpha) \;&\equiv\; \exp\{I_a\,\alpha_a\} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,|\,\boldsymbol{\alpha}\,|^{\,2k} \\
&+\; \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,|\,\boldsymbol{\alpha}\,|^{\,2k}\,I_a\,\alpha_a \\
&+\; \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+2)!}\,|\,\boldsymbol{\alpha}\,|^{\,2k}\,\Xi(\alpha) \\
&=\; \cos|\,\boldsymbol{\alpha}\,| + \frac{\sin|\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|}\,I_a\,\alpha_a + \frac{\Xi(\alpha)}{|\,\boldsymbol{\alpha}\,|^{\,2}}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\,|\,\boldsymbol{\alpha}\,|^{\,2k} \\
&=\; \cos|\,\boldsymbol{\alpha}\,| + \frac{\sin|\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|}\,I_a\,\alpha_a + \frac{1-\cos|\,\boldsymbol{\alpha}\,|}{|\,\boldsymbol{\alpha}\,|^{\,2}}\;\Xi(\alpha) \qquad (1.17)
\end{aligned}$$

Now the manifold of the canonical coordinates can be divided into two parts : an inside shell $0 \leq |\boldsymbol{\alpha}| < \pi$ and an outer region $\pi \leq |\boldsymbol{\alpha}| < 2\pi$. To each point in the inside shell we can assign a point in the outer region by means of the correspondence

$$\boldsymbol{\alpha}' = (-\boldsymbol{\alpha})\,\frac{2\pi - |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} \qquad 0 \leq |\boldsymbol{\alpha}| < \pi\,, \quad \pi \leq |\boldsymbol{\alpha}'| < 2\pi \tag{1.18}$$

The $SU(2)$ elements corresponding to $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are then related by

$$\boldsymbol{\tau}_F(\alpha') = -\,\boldsymbol{\tau}_F(\alpha)$$

All points on the boundary of the parameter space, a 2–sphere of radius 2π, correspond to the very same element *i.e.* $\boldsymbol{\tau}_F(\,|\boldsymbol{\alpha}| = 2\pi\,) = -\,\mathbf{1}$.

Conversely, it turns out that for any pair of canonical coordinates (α, α') connected by the relation (1.18) we have

$$\boldsymbol{\tau}_A(\alpha) = \boldsymbol{\tau}_A(\alpha')$$

As a consequence the adjoint exponential representation (1.17) is an irreducible and unitary representation of the rotation group that is not a faithful one on the whole domain of the canonical coordinates of the rotation group

$$D = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)\,|\,\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < (2\pi)^2\}$$

the adjoint $SO(3)$ representation being an homomorphism – the map $\boldsymbol{\tau}_A(\alpha) : D \to SO(3)$ is not iniective – at variance with respect to the fundamental $SU(2)$ representation which is faithful.

This implies in turn that, for $0 \leq |\boldsymbol{\alpha}| < 2\pi$, there are paths which are closed in the $SO(3)$ adjoint representation and which can not deformed into a point. Conversely, any closed path in the $SU(2)$ fundamental representation can always be deformed into a point. This means that the spinor representation of the rotation group is simply connected, though the vector representation is not.

Owing to this $SU(2)$ is said to be the *universal covering* of $SO(3)$.

It is worthwhile to remark that, on the one hand, had we used the eulerian angles (1.5) to parametrize the manifold of the proper rotation group $SO(3)$, then the irreducible, orthogonal adjoint representation $\boldsymbol{\tau}_A(\varphi, \theta, \psi)$ turns out to be manifestly faithful. On the other hand, the generic element of the irreducible and unitary fundamental spinor representation $\boldsymbol{\tau}_F(\varphi,\theta,\psi)$ can be expressed in terms of the eulerian angles as [5]

$$\boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi) =$$
$$\begin{pmatrix} \exp\{\mathrm{i}\,(\psi+\varphi)/2\}\cos\theta/2 & -\mathrm{i}\exp\{\mathrm{i}\,(\psi-\varphi)/2\}\sin\theta/2 \\ -\mathrm{i}\exp\{\mathrm{i}\,(\varphi-\psi)/2\}\sin\theta/2 & \exp\{-\mathrm{i}\,(\psi+\varphi)/2\}\cos\theta/2 \end{pmatrix}$$

so that for instance

$$\boldsymbol{\tau}_{\frac{1}{2}}(0,\theta,\psi) \;=\; -\,\boldsymbol{\tau}_{\frac{1}{2}}(2\pi,\theta,\psi)$$

Hence it follows that to any rotation $g(\varphi,\theta,\psi)$ there correspond two opposite matrices $\pm\,\boldsymbol{\tau}_{\frac{1}{2}}(\varphi,\theta,\psi)$.

Thus, in terms of the eulerian angles, the adjoint vector representation $\boldsymbol{\tau}_A(\varphi,\theta,\psi)$ appears to be a real, faithful, orthogonal, irreducible single-valued representation of the rotation group, whereas the fundamental spinor representation $\boldsymbol{\tau}_F(\varphi,\theta,\psi)$ turns out to be a unitary irreducible double-valued representation of the rotation group. Hence, the rotation group $SO(3)$ in the three dimensional space is not simply connected, its universal enveloping group being $SU(2)$.

- Unitary representations : as it is well known from the theory of the angular momentum in quantum mechanics, it turns out that all the irreducible, unitary, finite dimensional representations of the proper rotation group are labelled by their *weight*, which is a non–negative integer or half–integer number : namely,

$$\{\boldsymbol{\tau}_j \mid\ j = n/2\ (n = 0,1,2,\ldots)\} \qquad \dim\boldsymbol{\tau}_j = 2j+1$$

From the composition law of the angular momenta, it turns out that the product $\boldsymbol{\tau}_j \times \boldsymbol{\tau}_k$ of two irreducible unitary representations of the three dimensional rotation group of weights j and k contains just once each of the irreducible unitary representations

$$\boldsymbol{\tau}_i \qquad (i = |j-k|, |j-k|+1, \ldots, j+k-1, j+k)$$

Thus the following formula is valid

$$\boldsymbol{\tau}_j \times \boldsymbol{\tau}_k = \bigoplus_{i=|j-k|}^{j+k} \boldsymbol{\tau}_i \qquad \left(\forall\, j,k = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\right)$$

- Other important examples of Lie groups are :

    1. the full orthogonal groups $O(N)$ of the $N \times N$ orthogonal real matrices with determinant equal to ± 1 and their special or proper subgroups $SO(N)$ with unit determinant, the dimensions of which are $n = \frac{1}{2}\,N(N-1),;$
    2. the groups $U(N)$ of the $N \times N$ unitary complex matrices, the dimensions of which are $n = N^2$; in fact the generic $U(N)$ matrix depends upon $2N^2$ real parameters, but the request of unitarity entails N^2 real conditions so that the dimension of the Lie group $U(N)$ is $n = N^2$.
    3. the groups $SU(N)$ of the $N \times N$ special unitary complex matrices, *i.e.* with unit determinant, of dimensions $n = N^2 - 1$.

       Notice that $U(N)$ is homeomorphic to the product $SU(N) \times U(1)$.
    4. The special linear groups $SL(N, \mathbb{R})$ of the $N \times N$ real matrices of unit determinant, the dimensions of which are $n = N^2 - 1$;
    5. the special linear groups $SL(N, \mathbb{C})$ of the $N \times N$ complex matrices of unit determinant, the dimensions of which are $n = 2N^2 - 2$;

       The distinguished case of the six dimensional special linear group $SL(2, \mathbb{C})$ will be of particular relevance, as it turns out to be the universal covering spinorial group of the Lorentz group, *i.e.* it will play the same role in respect to the Lorentz group as the unitary group $SU(2)$ did regard to the three dimensional rotation group.

1.3 The Inhomogeneous Lorentz Group

1.3.1 The Lorentz Group

Consider a spacetime point specified in two inertial coordinate frames S and S', where S' moves with constant relative velocity $\mathbf{v}$ with respect to S. In S the spacetime point is labelled by (x, y, z, t) and in S' by (x', y', z', t'). The transformation that relates the two inertial coordinate frames is called a Lorentz transformation: according to the postulates of the special theory of relativity, it has the characteristic property ($c = 299\,792\,458$ m s^{-1} is the velocity of light in vacuum)

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

where we are assuming that the origins of both inertial frame coordinate systems do coincide for $t = t' = 0$ – for the moment we do not consider translations under which the relative distances remain invariant; if these are included one finds inhomogeneous Lorentz transformations, also called Poincaré transformations. We shall use the following standard notations and conventions [2] – sum over repeated indices is understood

$$x^{\mu} \equiv (x^0 = ct, x, y, z) = (x^0, \mathbf{x}) = (x^0, x^k) \qquad \mu = 0, 1, 2, 3 \quad k = 1, 2, 3$$

so that

$$x^2 \equiv g_{\mu\nu}\, x^{\mu} x^{\nu} = g_{\mu\nu}\, x'^{\mu} x'^{\nu} \equiv x'^2$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Owing to spacetime homogeneity and isotropy the Lorentz transformations are linear

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

which implies

$$g_{\rho\sigma} = g_{\mu\nu}\, \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\sigma} \tag{1.19}$$

or even in matrix notations

$$x^{\mu}\, x_{\mu} = x^{\top} \cdot g\, x = x'^{\mu}\, x'_{\mu} \qquad x' = \Lambda\, x \qquad g = \Lambda^{\top}\, g\, \Lambda$$

where $^{\top}$ denotes transposed matrix. Eq. (1.19) does indeed define the Lorentz group L as a group of 4×4 square–matrices acting upon Minkowski spacetime point column four vectors. As a matter of fact we have

1. composition law : $\Lambda\,,\ \Lambda' \in L \quad \Rightarrow \quad \Lambda \cdot \Lambda' = \Lambda'' \in L$
   matrix products : $(\Lambda''\,)^{\,\mu}{}_{\rho} = \Lambda^{\,\mu}{}_{\nu}\,(\Lambda')^{\,\nu}{}_{\rho}$
2. identity matrix : $\exists\,!\ \mathbf{I}$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. inverse matrix : from the defining relation (1.19) we have

$$\det g = \det\,(\Lambda^{\top}\,g\,\Lambda) \quad \Rightarrow \quad \det\Lambda = \pm 1$$

   and thereby $\exists\,!\ \Lambda^{-1} = g\cdot\Lambda^{\top}\cdot g \qquad \forall\Lambda\in L$

which means that L is a group of matrices that will be called the homogeneous full Lorentz group. From the relation

$$1 = g_{00} = g_{\mu\nu}\,\Lambda^{\,\mu}{}_{0}\,\Lambda^{\,\nu}{}_{0} = \left[\,\Lambda^{0}{}_{0}\,\right]^{\,2} - \Lambda^{\,k}{}_{0}\,\Lambda^{\,k}{}_{0} \quad \Rightarrow \quad \left|\,\Lambda^{0}{}_{0}\,\right| \geq 1$$

it follows that the homogeneous full Lorentz group splits into four categories called *connected components*

- proper orthochronus $\quad L_{+}^{\uparrow} = \{\Lambda\in L\,|\,\det\Lambda = 1 \ \cap\ \Lambda^{0}{}_{0} \geq 1\}$
- improper orthochronus $\quad L_{-}^{\uparrow} = \{\Lambda\in L\,|\,\det\Lambda = -1 \ \cap\ \Lambda^{0}{}_{0} \geq 1\}$
- proper non–orthochronus $\quad L_{+}^{\downarrow} = \{\Lambda\in L\,|\,\det\Lambda = 1 \ \cap\ \Lambda^{0}{}_{0} \leq -1\}$
- improper non–orthochronus $\quad L_{-}^{\downarrow} = \{\Lambda\in L\,|\,\det\Lambda = -1 \ \cap\ \Lambda^{0}{}_{0} \leq -1\}$

Among the four connected components of the homogeneous full Lorentz group there is only $L_{+}^{\uparrow}$ which is a subgroup, *i.e.* the component connected with the identity element, which is also called the *restricted* subgroup of the homogeneous full Lorentz group. Other quite common notations are as follows. The homogeneous full Lorentz group is also denoted by $O(1,3)$, the proper orthochronus component $L_{+}^{\uparrow}$ by $SO(1,3)^{+}$ or $O(1,3)_{+}^{+}$ and is also called the restricted homogeneous Lorentz group. The remaining connected components are also respectively denoted by

$$O(1,3)_{+}^{-} = L_{+}^{\downarrow} \qquad O(1,3)_{-}^{+} = L_{-}^{\uparrow} \qquad O(1,3)_{-}^{-} = L_{-}^{\downarrow}$$

Examples :

1. special Lorentz transformation, or even boost, in the OX direction with velocity $v > 0$ towards the positive OX axis

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cosh\eta = (1-\beta^2)^{-1/2} \qquad \sinh\eta = \beta\,(1-\beta^2)^{-1/2} \qquad \beta = \frac{v}{c}$$

2. spatial rotation around the OZ axis

$$\Lambda(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   boosts and rotations belong to the special homogeneous Lorentz group.

3. parity transformation with respect to the OY axis

$$\Lambda_P^y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   full spatial reflection or full parity transformation

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.20}$$

   spatial reflections or parity transforms belong to $L_-^{\uparrow} = O(1,3)_-^+$

4. time inversion or time reflection T transformation

$$\Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in\; L_-^{\downarrow} = O(1,3)_-^-$$

5. full inversion or PT transformation

$$\Lambda_{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;\in\; L_{+}^{\downarrow} = O(1,3)_{+}^{-}$$

Any Lorentz transformation can always been decomposed as the product of transformations of the four types belonging to the above described connected components. Since there are three independent rotations as well as three independent boosts, of for each spatial direction, the special homogeneous Lorentz transformations are described in terms of six parameters.

The homogeneous Lorentz group is a six dimensional Lie group so that each element can be labelled by six real parameters. For example, one can choose the three eulerian angles and the three ratios between the components of the relative velocity and the light velocity : namely,

$$g \in O(1,3) \;\rightarrow\; (\varphi, \theta, \psi, \beta_1, \beta_2, \beta_3) \qquad (\,\beta_k \;\equiv\; v_k/c\,)$$
$$0 \le \varphi \le 2\pi\,,\;\; 0 \le \theta < \pi\,,\;\; 0 \le \psi \le 2\pi\,,\;\; -1 < \beta_k < 1 \quad (\,k = 1,2,3\,)$$

The most suitable parametrization is in terms of the *canonical coordinates*

$$(\boldsymbol{\alpha}\,, \boldsymbol{\eta}\,) = (\alpha_1, \alpha_2, \alpha_3\,;\, \eta_1, \eta_2, \eta_3) \tag{1.21}$$

where the angles α_k $(k = 1,2,3)$ with $\boldsymbol{\alpha}^2 < (2\pi)^2$ are related to the spatial rotations around the orthogonal axes of the chosen inertial frame, whilst the hyperbolic arguments $\eta_k \equiv \mathrm{Arsh}\left(\beta_k\,(1-\beta_k^2)^{\,-1/2}\right)$ $(k = 1,2,3)$ with $\boldsymbol{\eta} \in \mathbb{R}^3$ are related to boosts along the spatial directions. More specifically

$$\Lambda(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$$

represents a rotation of the inertial reference frame in the counterclockwise sense of an angle α_1 around the OX axis whereas *e.g.*

$$\Lambda(\beta_3) = \begin{pmatrix} \cosh\eta_3 & 0 & 0 & -\sinh\eta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta_3 & 0 & 0 & \cosh\eta_3 \end{pmatrix}$$

corresponds to a boost with a *rapidity parameter*

$$\eta_3 = \mathrm{Arsh}\left(\beta_3\,(1-\beta_3^2)^{\,-1/2}\right) \tag{1.22}$$

associated to the OZ direction. Notice that the inverse transformations can be immediately obtained after sending $\alpha_k \mapsto -\alpha_k$ and $\eta_k \mapsto -\eta_k$. Since the domain of the canonical coordinates is the unbounded subset of $\mathbb{R}^6$

$$D \equiv \{(\boldsymbol{\alpha}\,,\boldsymbol{\eta}\,)\,|\,\boldsymbol{\alpha}^2 < (2\pi)^2\,,\ \boldsymbol{\eta} \in \mathbb{R}^3\}$$

it follows that the Lorentz group is non–compact.

It is also very convenient to write the elements of the special homogeneous Lorentz group in the exponential form and to introduce the *infinitesimal generators* according to the standard convention [5]

$$\Lambda(\alpha_1) = \exp\{\alpha_1 I_1\} \qquad \Lambda(\beta_3) = \exp\{\eta_3 J_3\}$$

where

$$I_1 \equiv \frac{d\Lambda}{d\alpha_1}\,(\alpha_1 = 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$J_3 \equiv \frac{d\Lambda}{d\beta_3}\,(\beta_3 = 0) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

et cetera . It is very important to gather that the infinitesimal generators of the space rotations are antihermitean $I_k^\dagger = -I_k$ $(k = 1, 2, 3)$ whereas the infinitesimal generators of the special Lorentz transformations turn out to be hermitean $J_k = J_k^\dagger$ $(k = 1, 2, 3)$. One can also check by direct inspection that the infinitesimal generators do fulfill the following commutation relations: namely,

$$[\,I_j\,I_k\,] = -\,\varepsilon_{jkl}\,I_l \qquad [\,J_j\,J_k\,] = \varepsilon_{jkl}\,I_l \qquad [\,I_j\,J_k\,] = -\,\varepsilon_{jkl}\,J_l \tag{1.23}$$

$$(\,j, k, l = 1, 2, 3)$$

The above commutation relations univocally specify the Lie algebra of the homogeneous Lorentz group.

Together with the infinitesimal generators I_k, J_k $(k = 1, 2, 3)$ it is very convenient to use the matrices

$$A_k \equiv \frac{1}{2}\,(I_k + \mathrm{i}\,J_k) \qquad B_k \equiv \frac{1}{2}\,(I_k - \mathrm{i}\,J_k) \qquad k = 1, 2, 3$$

It is worthwhile to remark that all the above matrices are antihermitean

$$A_j^\dagger = -\,A_j \qquad B_k^\dagger = -\,B_k \qquad (\,j, k = 1, 2, 3)$$

The commutation relations for these matrices have an especially simple form:

$$[\,A_j\,A_k\,] = -\,\varepsilon_{jkl}\,A_l \qquad [\,B_j\,B_k\,] = -\,\varepsilon_{jkl}\,B_l \qquad [\,A_j\,B_k\,] = 0$$

$$(\,j,k,l = 1,2,3)$$

which follow from the commutation relations of the infinitesimal generators. We stress that the commutation relations for the operators A_k $(k = 1,2,3)$ are the same as those for the generators of the three–dimensional rotation group $SO(3)$ and of its universal covering group $SU(2)$. This is also true for the operators B_k $(k = 1,2,3)$.

The infinite dimensional reducible unitary representations of the angular momentum Lie algebra are very well known. Actually, one can diagonalize simultaneously the positive semidefinite operators

$$\begin{aligned} A_j^\dagger\,A_j = A_j\,A_j^\dagger &= -\,A_j\,A_j = -\,\mathbf{A}^2 \\ B_k^\dagger\,B_k = B_k\,B_k^\dagger &= -\,B_k\,B_k = -\,\mathbf{B}^2 \end{aligned}$$

the spectral resolutions of which read

$$-\,\mathbf{A}^2 = \sum_m m\,(m+1)\,\widehat{P}_m \qquad -\,\mathbf{B}^2 = \sum_n n\,(n+1)\,\widehat{P}_n$$

$$\mathrm{tr}\ \widehat{P}_m = 2m+1 \qquad \mathrm{tr}\ \widehat{P}_n = 2n+1$$

$$m\,,n \;=\; 0\,,\tfrac{1}{2}\,,1\,,\tfrac{3}{2}\,,2\,,\tfrac{5}{2}\,,3\,,\ldots\ldots$$

Those operators are called the *Casimir's operators* of the Lorentz group, the latter ones being defined by the important property of commuting with all the infinitesimal generators of the group. It follows therefrom that

- all the irreducible finite dimensional representations of the Lorentz group are labelled by a pair (m,n) of non–negative integer or half–integer numbers

- each irreducible representation $\boldsymbol{\tau}_{m\,n}$ contains $(2m+1)\,(2n+1)$ linearly independent states characterized by the eigenvalues of *e.g.* A_3 and B_3 respectively

- we can identify $S_k = I_k = A_k + B_k$ $(k = 1,2,3)$ with the components of the *spin angular momentum* operator, so that we can say that $\boldsymbol{\tau}_{m\,n}$ is an irreducible representation of the Lorentz group of spin angular momentum $s \,=\, m+n$

- the lowest dimensional irreducible representations of the Lorentz group are: the unidimensional scalar representation $\boldsymbol{\tau}_{0\,0}$, the two dimensional left spinor Weyl representation $\boldsymbol{\tau}_{\frac{1}{2}\,0}$, the two dimensional right spinor Weyl representation $\boldsymbol{\tau}_{0\,\frac{1}{2}}$ (the handedness is conventional)

  Hermann Klaus Hugo Weyl
  `one of the most influential personalities for Mathematics in the first half of the XXth century`
  Elmshorn, Germany, 9.11.1885 – Zurich, CH, 8.12.1955
  Gruppentheorie und Quantenmechanik (1928)

- under a parity transform $\Lambda_P = \Lambda_P^{-1}$ we find

$$I_k \;\mapsto \Lambda_P^{-1}\, I_k\, \Lambda_P = I_k \qquad J_k \;\mapsto \Lambda_P^{-1}\, J_k\, \Lambda_P = -\, J_k \tag{1.24}$$

$$A_k \;\mapsto \Lambda_P^{-1}\, A_k\, \Lambda_P = B_k \qquad B_k \;\mapsto \Lambda_P^{-1}\, B_k\, \Lambda_P = A_k$$

  i.e. the two inequivalent irreducible Weyl's representations interchange under parity. Hence, to set up a spinor representation of the full Lorentz group out of the two Weyl's representations, one has to consider the direct sum

$$\boldsymbol{\tau}_D = \boldsymbol{\tau}_{\frac{1}{2}\,0} \oplus \boldsymbol{\tau}_{0\,\frac{1}{2}}$$

  which is a reducible four dimensional representation of the restricted Lorentz group and is called the *Dirac representation*

- all the other irreducible representations of higher dimensions can be obtained from the well known Clebsch–Gordan–Racah multiplication and decomposition rule

$$\boldsymbol{\tau}_{m\,n} \times \boldsymbol{\tau}_{p\,q} = \bigoplus_{r\,,\,s}\; \boldsymbol{\tau}_{r\,s}$$

$$|\,m-p\,| \le r \le m+p \qquad |\,n-q\,| \le s \le n+q$$

  In particular, the spin 1 representation does coincide with the above introduced four vector representation, which we actually used to define the homogeneous Lorentz group

$$\boldsymbol{\tau}_{\frac{1}{2}\,0} \times \boldsymbol{\tau}_{0\,\frac{1}{2}} = \boldsymbol{\tau}_{\frac{1}{2}\,\frac{1}{2}}$$

  The antisymmetric Maxwell's field strength $F_{\mu\nu}$ transforms according to the reducible parity symmetrical representation $\boldsymbol{\tau}_{1\,0} \oplus \boldsymbol{\tau}_{0\,1}$.

- All the irreducible finite dimensional representations of the Lorentz group are non–unitary. As a matter of fact we have *e.g.*

$$\boldsymbol{\tau}_{m\,n}(\beta_3) = \exp\{\,\beta_3 J_3\,\} = \exp\{\,-\,\mathrm{i}\beta_3(A_3 - B_3)\,\}$$

  and thereby

$$\boldsymbol{\tau}^{\dagger}_{m\,n}(\beta_3) = \boldsymbol{\tau}_{m\,n}(\beta_3)\ \neq\ \boldsymbol{\tau}^{-1}_{m\,n}(\beta_3) = \boldsymbol{\tau}_{m\,n}(-\beta_3)$$

  All the unitary irreducible representations of the Lorentz group are infinite dimensional and have been classified by

  I.M. Gel'fand and M.A. Naimark
  Unitary representations of the Lorentz group
  Izv. Akad. Nauk. SSSR, matem. **11**, 411, 1947.

1.3.2 Semisimple Groups

Lie groups and their corresponding Lie algebras can be divided into three main categories depending upon the presence or absence of some *invariant subgroups* and *invariant subalgebras*.

By its very definition, an invariant subgroup $H \subseteq G$ satisfies the following requirement : for any elements $g \in G$ and $h \in H$ there always exists an element $h' \in H$ such that

$$gh = h'g$$

Of course, any Lie group G has two trivial invariant subgroups, G itself and the unit element.

Concerning the infinitesimal operators $J_a \in \mathcal{G}$ $(a = 1, 2, \ldots n)$ and $T_b \in \mathcal{H}$ $(b = 1, 2, \ldots m \leq n)$ of the corresponding Lie algebra and subalgebra, we shall necessarily find

$$[\, J_a\, T_b\,] = C_{abc}\, T_c \qquad (a = 1, 2, \ldots n\,,\ b, c = 1, 2, \ldots m \leq n) \tag{1.25}$$

- Groups that do not possess any nontrivial invariant subgroup are called *simple*.

- A weaker requirement is that the group G had no nontrivial invariant *abelian* subgroups : in such a circumstance G is said to be *semisimple*.

  In this case it can be proved that any semisimple Lie group G is *locally isomorphic* to the *direct product* of mutually commuting simple non–abelian groups $G_1, G_2, \ldots, G_s$. This means that $\forall\, g_\alpha \in G_\alpha\,,\ g_\beta \in G_\beta$ we have

  $$G \doteq G_1 \times G_2 \times \cdots \times G_s$$

  $$g_\alpha\, g_\beta = g_\beta\, g_\alpha \qquad \alpha \neq \beta \qquad (\alpha, \beta = 1, 2, \ldots, s)$$

  where the symbol $\doteq$ means that the analytic isomorphism is true in some suitable neighborhoods of the unit elements of the involved Lie groups.

  For the corresponding Lie algebras we have that $\forall\, I_\alpha \in \mathcal{G}_\alpha\,,\ I_\beta \in \mathcal{G}_\beta$

  $$\mathcal{G} = \bigoplus_{\alpha=1}^{s} \mathcal{G}_\alpha \qquad [\, I_\alpha\, I_\beta\,] = 0 \qquad \alpha \neq \beta \qquad (\alpha, \beta = 1, 2, \ldots, s)$$

  An example of a semisimple group is $SO(4) \doteq SO(3) \times SO(3)$.

- Groups that do contain some invariant abelian nontrivial subgroups are said to be *non–semisimple*.

Such groups do not always factorize into the direct product of an abelian invariant subgroup and a semisimple group. As an example, let us consider the two dimensional euclidean group or inhomogeneous orthogonal group $IO(2)$, which is the group of the roto-translations in the plane without reflections with respect to an axis of the plane. We shall denote a translation by the symbol $T(\mathbf{a})$, where $\mathbf{a} = (a_x, a_y)$ is a general displacement of all points in the OXY plane. Evidently

$$T(\mathbf{a})\,T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b})$$

It is elementary to proof that every element g of the group $IO(2)$ can be represented in the form of a product of a rotation around an arbitrary point O of the plane and a certain translation :

$$g = T(\mathbf{a})\,R_O \qquad\qquad g \in IO(2)$$

It is also straightforward to prove the following identities

$$g\,T(\mathbf{a})\,g^{-1} = T(g\,\mathbf{a}) \qquad\qquad g \neq T(\mathbf{b}) \tag{1.26}$$

$$T(\mathbf{a})\,R_O\,T(-\,\mathbf{a}) = R_{O+\mathbf{a}} \tag{1.27}$$

where $g \in IO(2)$, $g\,\mathbf{a}$ is the point in the plane obtained from $\mathbf{a}$ by the displacement $g \in IO(2)$, R_O any rotation around the point O and $O + \mathbf{a}$ the point to which O moves owing to the translation of $\mathbf{a}$. Now, if we choose

$$g(\alpha) = \exp\{\alpha\, I_z\} \qquad 0 \le \alpha \le 2\pi \qquad T(\mathbf{a}) = \exp\{\mathbf{a}\cdot\mathbf{P}\}$$

where

$$I_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad\qquad g(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

from eq. (1.26) and for very small α, a_x, a_y we readily obtain the Lie algebra among the generators of $IO(2)$, *i.e.*

$$[\,P_x\,P_y\,] = 0 \qquad [\,I_z\,P_x\,] = P_y \qquad [\,I_z\,P_y\,] = -\,P_x$$

which precisely corresponds to the condition (1.25). Hence, translations do constitute an invariant abelian subgroup of $IO(2)$ and consequently $IO(2)$ is non–semisimple.

A very important quantity is the *Cartan–Killing* metric of a Lie group G , which is defined to be

$$g_{ab} \;\equiv\; C_{acd}\,C_{bdc} \qquad (a,b,c,d = 1,2,\ldots,n) \tag{1.28}$$

where C_{abc} are the structure constants of the Lie algebra $\mathcal{G}$. According to the correspondence (1.10) we can also write

$$\mathrm{tr}\,(A_a\,A_b) \;=\; g_{ab} \qquad (a,b=1,2,\ldots,n)$$

where $n\times n$ square matrices A_a $(a=1,2,\ldots,n)$ denote the generators in the adjoint representation. For instance, in the case of $SU(2)$ we find

$$g_{ab} = (-\,\varepsilon_{acd})(-\,\varepsilon_{bdc}) = -\,\varepsilon_{acd}\,\varepsilon_{bcd} \;=\; -\,2\,\delta_{ab}$$

It is important to remark that the Cartan–Killing metric of a Lie group G is a group-invariant. As a matter of fact, for any inner automorphism of the adjoint representation, see the definition (1.9), we find

$$\begin{aligned}
\mathrm{tr}\left(A_a(g)\,A_b(g)\right) &= \mathrm{tr}\left(T_A(g)\,A_a\,T_A^{-1}(g)\,T_A(g)\,A_b\,T_A^{-1}(g)\right) \\
&= \mathrm{tr}\left(T_A(g)\,A_a\,A_b\,T_A^{-1}(g)\right) = \mathrm{tr}\left(A_a\,A_b\,T_A^{-1}(g)\,T_A(g)\right) \\
&= \mathrm{tr}\,(A_a\,A_b) \;=\; g_{ab} \\
&\forall \;\; g\in G \qquad (a,b=1,2,\ldots,n)
\end{aligned} \tag{1.29}$$

where $T_A(g)$ $(g\in G)$ are the linear operators of the adjoint representation and I have made use of the *cyclicity property* of the trace operation, that is

$$\mathrm{tr}\,(A_1A_2\cdots A_n) = \mathrm{tr}\,(A_2\cdots A_nA_1) = \mathrm{tr}\,(A_nA_1A_2\cdots A_{n-1}) = \cdots$$

In particular, for an infinitesimal group transformation

$$T_A(g) = \mathbf{I}_A + \alpha_c\,A_c \qquad\qquad |\,\alpha_c\,| \ll 1$$

where $\mathbf{I}_A$ is the identity operator in the adjoint representation, the above qualities entail

$$\begin{aligned}
0 &= \delta\,g_{ab} = \alpha_c\,\mathrm{tr}\left([\,A_c\,A_a\,]\,A_b\right) + \mathrm{tr}\left(A_a[\,A_c\,A_b\,]\right)\alpha_c \\
&= \alpha_c\,C_{cad}\,\mathrm{tr}\,(A_d\,A_b) + C_{cbd}\,\mathrm{tr}\,(A_a\,A_d)\,\alpha_c \\
&= (C_{cad}\,g_{bd} + C_{cbd}\,g_{ad})\,\alpha_c = (C_{cab}+C_{cba})\,\alpha_c
\end{aligned} \tag{1.30}$$

whence

$$C_{cab} \;=\; -\,C_{cba} \tag{1.31}$$

which means antisymmetry of the structure constants also respect the last two indices. As a consequence the structure constants of any Lie group turn out to be *completely antisymmetric* with respect to the exchange of all three indices : namely, $C_{abc} = -\,C_{bac} = C_{bca} = -\,C_{cba}$.

The Cartan–Killing metric is non-singular, *i.e.* $\det \| g \| \neq 0$, if and only if the group is semisimple. Here $\| g \|$ indicates the $n \times n$ matrix of elements g_{ab} . For example we have

$$g_{ab} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{for } SU(2) \tag{1.32}$$

$$g_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{for } IO(2)$$

The Cartan–Killing metric is a real symmetric matrix : therefore, it can be diagonalized by means of an orthogonal transformation that reshuffles the infinitesimal operators.

1. If there are null eigenvalues in the diagonal form of the metric, then the Lie group is non–semisimple.

2. If the Cartan–Killing metric is negative definite, then the Lie group is compact.

3. If the Cartan–Killing metric has both positive and negative eigenvalues, as in the case of the homogeneous Lorentz group or the group $SL(2, \mathbb{R})$, then the Lie group is non–compact.

Since the Cartan–Killing metric is a non–singular $n \times n$ matrix for semisimple Lie groups it is possible to define its inverse matrix g^{ab} , which can be used to raise and lower the group indices $a, b, c, \ldots = 1, 2, \ldots, n$. For example, the Lie product, or commutator in the quantum physics terminology, of two generators in the representation R will be written as

$$[\, I_a^R \, I_b^R \,] = C_{abc}\, g^{cd}\, I_d^R$$

because contractions over lower group indices have to be performed by means of the inverse Cartan-Killing metric tensors.

For any semisimple Lie group G it is always possible to select a special element for any representation of its semisimple Lie algebra $\mathcal{G}$, that is

$$C_R \equiv g_{ab}\, I_R^a\, I_R^b = g^{ab}\, I_a^R\, I_b^R \qquad I_R^a \equiv g^{ab} I_b^R \qquad (a, b = 1, 2, \ldots, n) \tag{1.33}$$

the subscript R labelling some particular representation of the infinitesimal operators. The quadratic operator C_R is called the *Casimir's operator* of the representation R and turn out to be group invariant : namely,

$$T_R(g)\, C_R\, T_R^{-1}(g) = C_R \qquad \forall\, g \in G$$

or else

$$[\, T_R(g)\, C_R\,] = 0 \qquad\qquad \forall\, g \in G$$

`Proof` : first we notice that

$$\begin{aligned} [\, I_c^R\, C_R\,] &= g^{ab}\, [\, I_c^R\, (I_a^R\, I_b^R)\,] \\ &= g^{ab}\, [\, I_c^R\, I_a^R\,]\, I_b^R + g^{ab}\, I_a^R\, [\, I_c^R\, I_b^R\,] \\ &= g^{ab}\, C_{cad}\, g^{de}\, I_e^R\, I_b^R + g^{ab}\, I_a^R\, C_{cbd}\, g^{de}\, I_e^R \\ &= g^{ab}\, C_{cad}\, g^{de}\, I_e^R\, I_b^R + g^{ab}\, C_{cbd}\, g^{de}\, I_a^R\, I_e^R \end{aligned}$$

Reshuffling indices in the second addendum of the last line we obtain

$$[\, I_c^R\, C_R\,] \;=\; C_{cad}\left(g^{ab}\, g^{ed} + g^{ea}\, g^{db} \right) I_e^R\, I_b^R \tag{1.34}$$

and owing to complete antisymmetry of the structure constants with respect to all indices we eventually find

$$[\, I_c^R\, C_R\,] = 0$$

By iterating the above procedure it is immediate to verify that

$$[\, (\,\alpha_a\, I_a^R\,)^k\, C_R\,] = 0 \qquad \forall\, k \in \mathbb{N} \qquad \alpha_a \in \mathbb{R} \qquad (\, a = 1, 2, \ldots, n\,)$$

Hence, from the exponential representation,

$$\begin{aligned} T_R\,(g)\, C_R &= \exp\{\alpha_a\, I_a^R\}\, C_R = \sum_{k=0}^{\infty} \frac{1}{k!}\, (\,\alpha_a\, I_a^R\,)^k\, C_R \\ &= C_R \sum_{k=0}^{\infty} \frac{1}{k!}\, (\,\alpha_a\, I_a^R\,)^k \\ &= C_R \exp\{\alpha_a\, I_a^R\} = C_R\, T_R\,(g) \qquad \forall\, g \in G \end{aligned}$$

which completes the proof. □

The general structure of the Casimir's operators is strongly restricted by

- *Schur's lemma* : if a linear operator commutes with *every* element of any *irreducible* representation $\boldsymbol{\tau}(g)$ of a group, then it is proportional to the unit operator.

  For a Lie group this lemma may also be rephrased as follows.

- If a linear operator commutes with *all* the generators in an *irreducible* representation of a Lie algebra $\mathcal{G}$, then it must be proportional to the unit operator.

See [5] for the proof. According to Schur's lemma we come to the conclusion that in the irreducible representation labelled by R we have

$$C_R = d_R \, \mathbf{I}_R$$

where d_R is a number, depending upon the irreducible representation, which is called the *Dynkin's index*.

- For the adjoint representation we have

  $$\operatorname{tr} C_A = g^{ab} \operatorname{tr}(A_a A_b) = d_A \operatorname{tr} \mathbf{I}_A = n \, d_A = g^{ab} g_{ab} = n$$

  so that $d_A = 1$.

- In the case of the special unitary group $SU(2)$, *i.e.* in the case of the fundamental representation of the rotation group, from the very definition (1.13) and the Cartan-Killing metric (1.32), the Casimir's operator (1.33) reads – remember that $g^{ab} = -\frac{1}{2}\delta^{ab}$

  $$\operatorname{tr} C_F = 2\, d_F = -\tfrac{1}{2}\,\delta^{ab} \operatorname{tr}(\tau_a \tau_b) = -\tfrac{1}{2}\,\delta^{ab} \left(-\tfrac{1}{4}\right) 2\,\delta_{ab} = \tfrac{3}{4}$$

  so that $d_F = \frac{3}{8}$ and $d_A = 1$ for the rotation group.

- For any irreducible representation of the rotation group it is possible to show that

  $$C_R = \tfrac{1}{2} J(J+1)\, \mathbf{I}_R$$

  where J is the weight of the irreducible representation, which is related to the eigenvalue of the total angular momentum by $\mathbf{J}^2 = \hbar^2 J(J+1)$. Hence

  $$d_J = \tfrac{1}{2} J(J+1) \qquad \left(J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots\right)$$

  are the Dynkin's indices for the irreducible unitary representations of the rotation group.

- For a non–semisimple Lie group G, just like the Poincaré group for example, owing to the singular nature of the Cartan–Killing metric tensor, the Casimir operators in any irreducible representation R are just defined by the very requirement of being *invariant* under the whole group of linear transformations $T_R(g)$ $(g \in G)$: namely,

  $$C_R = T_R(g)\, C_R\, T_R^{\,-1}(g) \qquad \forall\, g \in G$$

1.3.3 The Poincaré Group

The quite general symmetry group of all the relativistic classical and quantum field theories obeying the special principle of relativity is the (restricted) inhomogeneous Lorentz group, also named the Poincaré group, under which

$$x'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\,\nu} + a^{\,\mu}$$

where $a^{\,\mu}$ is an arbitrary constant four vector. Hence the Poincaré group is a non–semisimple ten parameters Lie group, the canonical coordinates of which, in accordance with (1.21), can be identified with

$$(\boldsymbol{\alpha}\,,\boldsymbol{\eta}\,,a\,) = \Big(\alpha_1,\alpha_2,\alpha_3\,;\,\eta_1,\eta_2,\eta_3\,;\,a^0,a^1,a^2,a^3\Big)$$

The spacetime translations $T(a)$ do constitute an abelian four parameters subgroup and fulfill

$$T(a)\,T(b) = T(a+b) = T(b)\,T(a)$$

However, spacetime translations do not commute with the Lorentz group elements. Consider in fact two Poincaré transformations with parameters (Λ, a) and (Λ', a') so that

$$x^{\mu} \quad\mapsto\quad \Lambda^{\mu}{}_{\nu}\, x^{\,\nu} + a^{\,\mu} \quad\mapsto\quad \Lambda'^{\mu}_{\ \rho}\Big(\Lambda^{\rho}{}_{\nu}\, x^{\,\nu} + a^{\,\rho}\Big) + a'^{\,\mu}$$

$$a^{\,\mu} \quad\mapsto\quad \Lambda'^{\mu}_{\ \rho}\, a^{\,\rho} + a'^{\,\mu}$$

whence we see that the translation parameters get changed under a Lorentz transformation. Owing to this feature, which is called the *soldering* between the Lorentz transformations and the spacetime translations, the Poincaré group is said to be a *semidirect* product of the Lorentz group and the spacetime translation abelian group.

The generators of the spacetime translations are the components of the four gradient operator. As a matter of fact, for any analytic real function $f : \mathcal{M} \to \mathbb{R}$ we have

$$T(a)\,f(x) = \exp\left\{a^{\,\mu}\partial_{\mu}\right\} f(x) = f(x+a)$$

$$[\,\partial_{\mu}\,,\,\partial_{\nu}\,] \equiv \partial_{\mu}\,\partial_{\nu} - \partial_{\nu}\,\partial_{\mu} = 0$$

Thus, the infinitesimal operators of the spacetime translations turn out to be differential operators acting of the infinite dimensional space $C_{\infty}(\mathcal{M})$ of the analytic functions on the Minkowski's spacetime.

It is necessary to obtain an infinite dimensional representation of the generators of the Lorentz group acting on the very same functional space. Introduce the six antihermitean differential operators

$$\ell_{\mu\nu} \equiv g_{\mu\rho}\, x^{\rho}\, \partial_{\nu} - g_{\nu\rho}\, x^{\rho}\, \partial_{\mu} \equiv x_{\mu}\, \partial_{\nu} - x_{\nu}\, \partial_{\mu} \tag{1.35}$$

$$\ell_{\mu\nu} = -\,\ell_{\nu\mu} = -\,\ell^{\dagger}_{\mu\nu}$$

By direct inspection it is straightforward to verify the commutation relations

$$[\,\ell_{\mu\nu}\,,\,\ell_{\rho\sigma}\,] = -\,g_{\mu\rho}\,\ell_{\nu\sigma} + \,g_{\mu\sigma}\,\ell_{\nu\rho} - \,g_{\nu\sigma}\,\ell_{\mu\rho} + \,g_{\nu\rho}\,\ell_{\mu\sigma} \tag{1.36}$$

so that, if we make the correspondences

$$I_j \;\leftrightarrow\; \tfrac{1}{2}\,\varepsilon_{\,jkl}\,\ell_{kl} \qquad J_k \;\leftrightarrow\; \ell_{0k} \qquad (\,j,k,l = 1,2,3\,)$$

it can be readily checked that the above commutation relations among the differential operators $\ell_{\mu\nu}$ do realize an infinite dimensional representation of the Lie algebra (1.23) of the Lorentz group. Nonetheless, it is important to remark that all six generators in the above infinite dimensional representation of the Lorentz group are antihermitean, at variance with the infinitesimal operators of all the finite dimensional irreducible representations, in which only the three generators of the rotation subgroup are antihermitean.

Moreover we find

$$[\,\ell_{\mu\nu}\,,\,\partial_{\rho}\,] = -\,g_{\mu\rho}\,\partial_{\nu} + g_{\nu\rho}\,\partial_{\mu}$$

hence we eventually come to the infinite dimensional representation of the Lie algebra of the Poincaré group

$$\begin{aligned} &[\,\partial_{\mu}\,,\,\partial_{\nu}\,] = 0 \qquad\qquad [\,\ell_{\mu\nu}\,,\,\partial_{\rho}\,] = -\,g_{\mu\rho}\,\partial_{\nu} + g_{\nu\rho}\,\partial_{\mu} \\ &[\,\ell_{\mu\nu}\,,\,\ell_{\rho\sigma}\,] = -\,g_{\mu\rho}\,\ell_{\nu\sigma} + \,g_{\mu\sigma}\,\ell_{\nu\rho} - \,g_{\nu\sigma}\,\ell_{\mu\rho} + \,g_{\nu\rho}\,\ell_{\mu\sigma} \end{aligned} \tag{1.37}$$

However, in view of the applications to quantum mechanics and quantum field theory, it is convenient and customary to introduce a set of *hermitean generators* of the Poincaré group: namely,

$$P_{\mu} \equiv \mathrm{i}\,\partial_{\mu} = (\mathrm{i}\,\partial_0\,,\,\mathrm{i}\,\boldsymbol{\nabla}) \tag{1.38}$$

$$L_{\mu\nu} \equiv \mathrm{i}\,\ell_{\mu\nu} = x_{\mu}\,P_{\nu} - x_{\nu}\,P_{\mu} \tag{1.39}$$

and the corresponding Lie algebra

$$[\,P_{\mu}\,,\,P_{\nu}\,] = 0 \qquad [\,L_{\mu\nu}\,,\,P_{\rho}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,P_{\nu} + \mathrm{i}\,g_{\nu\rho}\,P_{\mu}$$

$$[\,L_{\mu\nu}\,,\,L_{\rho\sigma}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,L_{\nu\sigma} + \mathrm{i}\,g_{\mu\sigma}\,L_{\nu\rho} - \mathrm{i}\,g_{\nu\sigma}\,L_{\mu\rho} + \mathrm{i}\,g_{\nu\rho}\,L_{\mu\sigma} \tag{1.40}$$

One can easily recognize that the six hermitean differential operators $L_{\mu\nu}$ do actually constitute the relativistic generalization of the orbital angular momentum operator of non–relativistic quantum mechanics. We find indeed

$$\mathbf{L} = (L_{23}, L_{31}, L_{12}) = \mathbf{r} \times (-\mathrm{i}\,\boldsymbol{\nabla}) \qquad [\,L^{\jmath}\,,\,L^{k}\,] = \mathrm{i}\,\varepsilon^{\jmath k\ell}\,L^{\ell}$$

On the other hand, it turns out that the most general *infinite dimensional representation* of the Poincaré Lie algebra is given by

$$[\,P_{\mu}\,,\,P_{\nu}\,] = 0 \qquad [\,M_{\mu\nu}\,,\,P_{\rho}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,P_{\nu} + \mathrm{i}\,g_{\nu\rho}\,P_{\mu}$$
$$[\,M_{\mu\nu}\,,\,M_{\rho\sigma}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,M_{\nu\sigma} + \mathrm{i}\,g_{\mu\sigma}\,M_{\nu\rho} - \mathrm{i}\,g_{\nu\sigma}\,M_{\mu\rho} + \mathrm{i}\,g_{\nu\rho}\,M_{\mu\sigma}$$

where

$$M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu} \tag{1.41}$$

in which the *relativistic spin angular momentum* operator $S_{\mu\nu}$ must satisfy

$$[\,P_{\mu}\,,\,S_{\nu\rho}\,] = 0 = [\,S_{\mu\nu}\,,\,L_{\rho\sigma}\,] \tag{1.42}$$

$$[\,S_{\mu\nu}\,,\,S_{\rho\sigma}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,S_{\nu\sigma} + \mathrm{i}\,g_{\mu\sigma}\,S_{\nu\rho} - \mathrm{i}\,g_{\nu\sigma}\,S_{\mu\rho} + \mathrm{i}\,g_{\nu\rho}\,S_{\mu\sigma} \tag{1.43}$$

Also the above construction is nothing but the relativistic generalization of the most general representation for the Lie algebra of the three-dimensional euclidean group $IO(3)$ in the non–relativistic quantum mechanics

$$\mathbf{p} = -\,\mathrm{i}\hbar\,\boldsymbol{\nabla} \qquad \mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{r} \times \mathbf{p} + \mathbf{S}$$

where $\mathbf{p}, \mathbf{L}, \mathbf{S}$ are respectively the self–adjoint operators of the momentum, orbital angular momentum and spin angular momentum of a point–particle with spin.

Since the translation infinitesimal operators P_{μ} do evidently constitute an abelian invariant subalgebra, the restricted Poincaré group turns out to be a non–semisimple and non–compact Lie group, the diagonalization of the Cartan–Killing ten–by–ten real symmetric matrix leading to three positive, three negative and four null eigenvalues.

In order to single out the Casimir operators of the Poincaré group, it is useful to introduce the *Pauli-Lubanski* operator

$$W^{\mu} \;\equiv\; \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,P_{\nu}\,M_{\rho\sigma} \;=\; \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,P_{\nu}\,S_{\rho\sigma} \tag{1.44}$$

as well as the *Shirkov* operator

$$V^{\mu} \equiv M^{\mu\rho} P_{\rho}$$

which satisfy the relationships

$$W_{\mu} P^{\mu} = 0 = V_{\mu} P^{\mu}$$

$$P^{2} M_{\mu\nu} = V_{\mu} P_{\nu} - V_{\nu} P_{\mu} - \varepsilon_{\mu\nu\rho\sigma} W^{\rho} P^{\sigma}$$

It is immediate to check that the following commutation relations hold true

$$[W_{\mu}, P_{\nu}] = 0 \qquad [M_{\mu\nu}, W_{\rho}] = -\mathrm{i}\, g_{\mu\rho} W_{\nu} + \mathrm{i}\, g_{\nu\rho} W_{\mu}$$

so that we can readily recognize the two Casimir's operators of the Poincaré group to be the scalar and pseudoscalar invariants

$$C_m = P^{2} = P^{\mu} P_{\mu} \qquad C_s = W^{2} = W^{\mu} W_{\mu} \tag{1.45}$$

which are respectively said the mass and the spin operators. Now, according to Wigner's theorem [6]

Eugene Paul Wigner (Budapest 17.11.1902 – Princeton 1.1.1995) *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrick Vieweg und Sohn, Braunschweig, Deutschland, 1931, pp. 251–254, *Group Theory and its Application to the Quantum Theory of Atomic Spectra*, Academic Press Inc., New York, 1959, pp. 233–236

any symmetry transformation in quantum mechanics must be realized only by means of unitary or antiunitary operators. The representation theory for the Poincaré group has been worked out by Bargmann and Wigner [7]. The result is that *all* the unitary irreducible representations of the Poincaré group have been classified and fall into four classes.

1. The eigenvalue of the Casimir's operator $C_m \equiv m^2$ is real and positive, while the eigenvalues of the Casimir's operator $C_s \equiv -m^2 s(s+1)$, where s is the spin, do assume discrete values $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

   Such a kind of unitary irreducible representations $\boldsymbol{\tau}_{m,s}$ are labeled by the rest mass $m > 0$ and the spin s. The states belonging to this kind of unitary irreducible representations are distinguished, for instance, by *e.g.* the component of the spin along the OZ axis

   $$s_z = -s, -s+1, \ldots, s-1, s\,; \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

and by the continuous eigenvales of the spatial momentum $\mathbf{p}$, so that $p_0^2 = \mathbf{p}^2 + m^2$: namely,

$$|\, m, s\,;\, \mathbf{p}, s_z\,\rangle \qquad m > 0 \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\mathbf{p} \in \mathbb{R}^3 \qquad s_z = -s, -s+1, \ldots, s-1, s$$

Physically, these states will describe some elementary particle of rest mass m, spin s, momentum $\mathbf{p}$ and spin projection s_z along the OZ axis. Massive particles of spin s are described by wave fields which correspond to $2s+1$ real functions on Minkowski's spacetime.

2. The eigenvalues of both the Casimir's operators vanish, *i.e.* $P^2 = 0$ and $W^2 = 0$, and since $P^\mu W_\mu = 0$ it follows that W_μ and P_μ are light–like and proportional, the constant of proportionality being named the *helicity*, which is equal to $\pm\, s$, where, again, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ is the spin of the representation.

   The states belonging to this kind of unitary irreducible representations $\boldsymbol{\tau}_{0\,,\,s}$ are distinguished by *e.g.* two possible values of the helicity $\lambda = \pm s$ and by the continuous eigenvales of the momentum $\mathbf{p}$, so that $p_0^2 = \mathbf{p}^2$: namely,

$$|\, s\,;\, \mathbf{p}, \lambda\,\rangle \qquad m = 0 \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\mathbf{p} \in \mathbb{R}^3 \qquad \lambda = \pm\, s$$

   Thus, this kind of unitary irreducible representations of the Poincaré group will correspond to the *massless* particles that, for $s \neq 0$, are described by two independent real functions on the Minkowski's spacetime. Examples of particles falling in this category are the Maxwell's photons with spin 1 and two polarizations, the massless left–handed neutrinos with helicity $s = \frac{1}{2}$ and the right–handed antineutrinos of helicity $s = -\frac{1}{2}$, maybe the graviton with spin 2 and two polarizations.

3. The eigenvalue of the Casimir's mass operator is zero, $P^2 = 0$, while that one of the Casimir's spin operator is continuous, *i.e.* $W^2 = -w^2$ where $w > 0$. Thus, this kind of unitary irreducible representations $\boldsymbol{\tau}_{0\,,\,w}$ of the Poincaré group would correspond to elementary particles of zero rest mass, with an infinite and continuous number of polarizations. These objects have never been detected in Nature.

4. There are also *tachyon–like* unitary irreducible representations which $P^\mu P_\mu < 0$ which are not physical as they drive to acausality.

There are further irreducible representations of the Poincaré group but they are neither unitary nor antiunitary. As already remarked, Wigner's theorem [6] generally states that all symmetry transformations – just like Poincaré transformations – in quantum mechanics can be consistently realized *solely* by means of some unitary or antiunitary operators.

- *All the unitary representations of the Poincaré group turn out to be infinite dimensional*, corresponding to particle states with unbounded momenta $\mathbf{p} \in \mathbb{R}^3$.

- *Elementary particles* correspond to the *irreducible representations*, the reducible representations being naturally associated to the *composite objects*. For example, the six massive quarks with spin $s = \frac{1}{2}$ and with fractional electric charge $q = \frac{2}{3}$ or $q = -\frac{1}{3}$ are considered as truly elementary particles, although not directly detectable because of the dynamical confinement mechanism provided by Quantum Chromo Dynamics (QCD), while hadrons, like the nucleons and π mesons, are understood as composite objects.

- As already seen, all finite dimensional irreducible representations of the Lorentz group are nonunitary.

- The *relativistic quantum wave fields* on the Minkowski spacetime do constitute the *only available way* to actually implement the unitary irreducible representations of the Poincaré group, that will be therof associated to the elementary particles obeying the general principles of the quantum mechanics and of the special theory of relativity.

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Chapter 2

The Action Functional

2.1 The Classical Relativistic Wave Fields

2.1.1 Field Variations

As we shall see in the next chapters, in order to build up explicitly the infinite dimensional Hilbert spaces that carry on all the irreducible unitary representations of the Poincaré group, which describe the quantum states of the elementary particles we detect in Nature, we will perform the canonical quantization of some classical mechanical systems with an infinite number of degrees of freedom. These mechanical systems consist in a collection of real or complex functions defined on the Minkowski's spacetime $\mathcal{M}$

$$u_A(x) \,:\, \mathcal{M} \longrightarrow \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases} \qquad (A = 1, 2, \ldots, N)$$

with a well–defined transformation law under the action of the Poincaré group $\mathfrak{P}$. We shall call these systems the *classical relativistic wave fields*. More specifically, if we denote the elements of the Poincaré group by $g = (\Lambda, a) \in \mathfrak{P}$, where $\Lambda \in L$ is an element of the Lorentz group while a^{μ} specify a translation in the Minkowski's 4–dimensional spacetime $\mathcal{M}$, we have

$$\begin{aligned} u_A(x) \stackrel{g}{\longmapsto} u_A'(x') &\equiv u_A'(\Lambda x + a) \\ u_A'(x') &= [\,T(\Lambda)\,]_{AB}\, u_B(x) \\ &= [\,T(\Lambda)\,]_{AB}\, u_B\Big(\Lambda^{-1}(x' - a)\Big) \end{aligned} \tag{2.1}$$

$$(\Lambda, a) \in \mathfrak{P} \qquad (A, B = 1, 2, \ldots, N)$$

where $T(\Lambda)$ are the operators of a representation of the Lorentz group of finite dimensions N. This means that, if the collection of the wave field functions at the point $P \in \mathcal{M}$ of coordinates x^{μ} is given by $u_A(x)$ $(A = 1, 2, \ldots, N)$ in a certain inertial frame S , then in the new inertial frame S' , related to S by the Poincaré transformation $(\Lambda, a) \in \mathfrak{P}$, the spacetime coordinates of P will be changed to $x' = \Lambda x + a$ and contextually the wave field functions will be reshuffled as $u'_A(x')$ $(A = 1, 2, \ldots, N)$ because the functional relationships will be in general frame dependent.

We can always represent the collection of the classical relativistic wave field functions as an N–component column vector

$$u\,(x) = \left(\begin{array}{c} u_1(x) \\ u_2(x) \\ \vdots \\ u_{N-1}(x) \\ u_N(x) \end{array}\right) \qquad x \in \mathcal{M}$$

Then we can suitably introduce some finite quantities which are said to be the total variation, the local variation and the differential of the classical relativistic wave field $u\,(x)$ according to

$$u'\,(x') - u\,(x) \qquad \text{total variation} \tag{2.2}$$

$$u'\,(x') - u\,(x') \qquad \text{local variation} \tag{2.3}$$

$$u\,(x') - u\,(x) \qquad \text{differential variation} \tag{2.4}$$

so that we can write

$$u'\,(x') - u\,(x) = [\,u'\,(x') - u\,(x')\,] + [\,u\,(x') - u\,(x)\,]$$

that is the total variation is equal to the sum of the local and of the differential variations for any classical relativistic wave field.

Consider a first order infinitesimal (passive) Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu\rho}\,g_{\rho\nu} \qquad |\,\varepsilon^{\mu}{}_{\nu}\,| \ll 1 \tag{2.5}$$

On the one hand, from the defining relation (1.19) of the Lorentz matrices we obtain

$$0 = g_{\mu\rho}\,\varepsilon^{\rho}{}_{\nu} + g_{\nu\rho}\,\varepsilon^{\rho}{}_{\mu} \quad \Leftrightarrow \quad \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0$$

that is, the infinitesimal parameters $\varepsilon_{\mu\nu}$ constitute an antisymmetric matrix with six independent entries.

On the other hand, from the exponential formula (1.12) for the Lorentz matrices we can write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\alpha_k\,(I_k)^{\mu}{}_{\nu} + \delta\beta_k\,(J_k)^{\mu}{}_{\nu} \tag{2.6}$$

$$|\,\delta\alpha_k\,| \ll 1 \qquad |\,\delta\beta_k\,| \ll 1 \qquad (\,k = 1,2,3\,)$$

If we set

$$\delta\alpha_j = -\,\tfrac{1}{2}\,\varepsilon_{jkl}\,\delta\,\omega^{\,kl} \qquad \delta\,\omega^{\,kl} \;=\; \delta\alpha_j\,\varepsilon^{\,jkl} \;=\; \delta\,\omega_{\,kl} \;=\; -\,\delta\alpha_j\,\varepsilon_{\,jkl}$$

$$\delta\beta_k = \delta\,\omega_{\,k0} = \delta\,\omega^{\,0k}$$

$$\omega_{\,\rho\sigma} + \omega_{\,\sigma\rho} = 0$$

where ε_{jkl} is the Levi-Civita symbol, totally antisymmetric in all of its three indices and normalized as $\varepsilon_{123} \;=\; +\,1 \;=\; -\,\varepsilon^{123}$, together with

$$I_j = \tfrac{1}{2}\,\mathrm{i}\,\varepsilon_{jkl}\,S_{kl} \qquad J_k = -\,\mathrm{i}\,S_{\,0k} \qquad S_{\,\rho\sigma} + S_{\,\sigma\rho} = 0 \tag{2.7}$$

then we readily obtain

$$\Lambda^{\mu}_{\;\;\nu} = \delta^{\,\mu}_{\;\;\nu} - \tfrac{1}{2}\,\mathrm{i}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\,\rho\sigma})^{\,\mu}_{\;\;\nu} \tag{2.8}$$

For example, a passive rotation around the OZ axis corresponds to

$$\begin{aligned}
\alpha_1 = \alpha_2 = 0\,, \quad \alpha_3 = -\,\delta\,\omega^{\,12} = -\,\varepsilon_{12} \\
\Lambda^{\mu}_{\;\;\nu} = \delta^{\,\mu}_{\;\;\nu} + \delta\,\omega^{12}\,(\delta^{\,\mu}_{\;\;1}\,g_{\,2\nu} - \delta^{\,\mu}_{\;\;2}\,g_{\,1\nu})
\end{aligned}$$

$$\Lambda(\alpha_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha_3 & 0 \\ 0 & -\,\alpha_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

Moreover, a passive boost along the OX axis is described by

$$\begin{aligned}
\beta_1 = \delta\,\omega^{\,01}\,, \qquad \beta_2 = \beta_3 = 0 \\
\Lambda^{\mu}_{\;\;\nu} = \delta^{\,\mu}_{\;\;\nu} + \delta\,\omega^{\,01}\,(\delta^{\,\mu}_{\;\;0}\,g_{\,1\nu} - \delta^{\,\mu}_{\;\;1}\,g_{\,0\nu})
\end{aligned}$$

$$\Lambda(\beta_1) = \begin{pmatrix} 1 & -\beta_1 & 0 & 0 \\ -\beta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.10}$$

A comparison between the expressions (2.5) and (2.8) yields

$$\begin{aligned}
\varepsilon^{\,\mu\lambda}\,g_{\,\lambda\nu} &= -\,\varepsilon^{\,\rho\sigma}\,\delta^{\,\mu}_{\;\;\sigma}\,g_{\,\rho\nu} \\
&= \tfrac{1}{2}\,\varepsilon^{\,\rho\sigma}\,(\delta^{\,\mu}_{\;\;\rho}\,g_{\,\sigma\nu} - \delta^{\,\mu}_{\;\;\sigma}\,g_{\,\rho\nu}) \\
&= -\,\tfrac{1}{2}\,\mathrm{i}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\,\rho\sigma})^{\,\mu}_{\;\;\nu}
\end{aligned} \tag{2.11}$$

Hence

$$\varepsilon^{\,\rho\sigma} = \delta\,\omega^{\,\rho\sigma} \qquad (S_{\,\rho\sigma})^{\,\mu}_{\;\;\nu} = \mathrm{i}\,\left(\delta^{\,\mu}_{\;\;\rho}\,g_{\,\sigma\nu} - \delta^{\,\mu}_{\;\;\sigma}\,g_{\,\rho\nu}\right) \tag{2.12}$$

The six matrices $S_{\rho\sigma}$ realize the relativistic spin angular momentum tensor of the irreducible vector representation $\boldsymbol{\tau}_{\frac{1}{2}\frac{1}{2}}$ of the Lorentz group. By the way, it is worthwhile to remark that the components S_{jk} are hermitean matrices and generate rotations, while the components S_{0k} are antihermitean matrices and generate boosts.

Turning back to the three kinds of infinitesimal variation we can write

$$x'^{\,\mu} \approx x^{\mu} + \delta x^{\mu} = x^{\mu} + \delta\omega^{\mu\nu} g_{\nu\rho} x^{\rho} + \delta\omega^{\mu} \tag{2.13}$$

$$\delta\omega^{\mu\nu} \;+\; \delta\omega^{\nu\mu} = 0 \qquad |\,\delta\omega^{\mu\nu}\,| \ll 1 \qquad |\,\delta\omega^{\mu}\,| \ll |\,x^{\mu}\,|$$

$$\begin{aligned}
\Delta u(x) \;&\equiv\; u'(x+\delta x) - u(x) \\
&\equiv\; [\,u'(x+\delta x) - u(x+\delta x)\,] + [\,u(x+\delta x) - u(x)\,] \\
&\equiv\; \delta u(x+\delta x) + d u(x) \\
&=\; \delta u(x) + \delta x^{\mu}\,\partial_{\mu}\,\delta u(x) + \;\cdots + d u(x) \\
&=\; \delta u(x) + \delta x^{\mu}\partial_{\mu} u(x) + O\,(\delta u\,\delta x)
\end{aligned} \tag{2.14}$$

so that we can safely write the suggestive symbolic relation

$$\Delta \;=\; \delta \;+\; d \;=\; \delta \;+\; \delta x^{\mu}\,\partial_{\mu} \tag{2.15}$$

among the first order infinitesimal variations together with

$$\Delta u(x) \approx \delta u(x) + d u(x) = \delta u(x) + \delta x \cdot \partial\, u(x) \tag{2.16}$$

It is important to remark that, by definition, the local variations do commute with the four gradient differential operator

$$\partial_{\mu}\,\delta u(x) = \delta\,\partial_{\mu} u(x) \quad\Leftrightarrow\quad [\,\delta\,,\,\partial_{\mu}\,] = 0 \tag{2.17}$$

Notice that the infinitesimal form of the Poincaré transformations for the spacetime coordinates can also be written in terms of the generators (1.39), that is

$$\delta x^{\mu} = \tfrac{1}{2}\,\delta\omega^{\rho\sigma} L_{\rho\sigma}\, x^{\mu} - \mathrm{i}\,\delta\omega^{\rho} P_{\rho}\, x^{\mu}$$

Here below we shall analyse the most relevant cases.

2.1.2 The Scalar and Vector Fields

1. Scalar field: the simplest case is that of a single invariant real function

$$\phi \,:\, \mathcal{M} \longrightarrow \mathbb{R} \qquad \phi'(x') = \phi(x) \tag{2.18}$$

so that

$$\Delta\,\phi\,(x) = 0 \quad\Leftrightarrow\quad \delta\,\phi\,(x) = -\,d\,\phi\,(x) = -\,\delta\,x\cdot\partial\,\phi\,(x)$$

From the infinitesimal change (2.13) we get the local variation

$$\begin{aligned}\delta\,\phi\,(x) &= -\,\delta\,\omega^{\,\mu\nu}\,g_{\,\nu\rho}\,x^{\,\rho}\,\partial_\mu\,\phi(x) -\,\delta\,\omega^{\,\mu}\,\partial_\mu\,\phi(x)\\ &= -\,\tfrac{1}{2}\,\mathrm{i}\,\delta\,\omega^{\,\mu\nu}\,L_{\,\mu\nu}\,\phi\,(x) + \mathrm{i}\,\delta\,\omega^{\,\mu}\,P_\mu\,\phi\,(x)\end{aligned}\qquad(2.19)$$

where, see the definition (1.39),

$$P_\mu = \mathrm{i}\,\partial_\mu \qquad L_{\,\mu\nu} \equiv x_{\,\mu}\,P_\nu - x_{\,\nu}\,P_\mu$$

whence it follows that for a scalar field we find by definition

$$M_{\mu\nu}\;\phi\,(x) \equiv L_{\,\mu\nu}\;\phi\,(x) \quad\Leftrightarrow\quad S_{\,\mu\nu}\;\phi\,(x) \equiv 0$$

i.e. the scalar field carries relativistic orbital angular momentum but not relativistic spin angular momentum.

It is worthwhile to consider also the *pseudoscalar* field, which are odd with respect to improper orthochronus Lorentz transformations, *i.e.*

$$\widetilde{\phi}^{\,\prime}\,(\Lambda x + a) = (\det\Lambda)\,\widetilde{\phi}\,(x) = -\,\widetilde{\phi}\,(x) \qquad \forall\,\Lambda\in L_{-}^{\uparrow} \qquad(2.20)$$

Complex scalar (pseudoscalar) field are complex invariant functions with scalar (pseudoscalar) real and imaginary parts.

2. Vector field : a relativistic (covariant) vector wave field is defined by the transformation law under the Poicaré group that reads

$$V_\mu^{\,\prime}\,(x^{\,\prime}) = V_\mu^{\,\prime}\,(\Lambda x + a) \equiv \Lambda_{\,\mu}^{\,\nu}\,V_\nu\,(x) \qquad(2.21)$$

The above transformation law can be obviously generalized to the aim of defining the arbitrary relativistic tensor wave fields of any rank with many contravariant and covariant Minkowski's indices

$$\begin{aligned}T^{\,\prime\,\alpha\beta\ldots}_{\mu\nu\ldots}\,(x^{\,\prime}) &= T^{\,\prime\,\alpha\beta\ldots}_{\mu\nu\ldots}\,(\Lambda x + a)\\ &\equiv \Lambda_\mu^{\,\rho}\,\Lambda_\nu^{\,\sigma}\,\ldots\,\Lambda_\lambda^{\,\alpha}\,\Lambda_\kappa^{\,\beta}\,\ldots\,T^{\,\lambda\kappa\ldots}_{\rho\sigma\ldots}\,(x)\end{aligned}\qquad(2.22)$$

It follows therefrom that the components of a vector field transform according to the irreducible vector representation $\boldsymbol{\tau}_{\frac{1}{2}\,\frac{1}{2}}$ of the Lorentz group. For infinitesimal Lorentz transformations we can write

$$\Lambda^{\,\mu}_{\;\;\nu} = \delta^{\,\mu}_{\;\;\nu} + \delta\omega^{\,\mu\rho}\,g_{\,\rho\nu} \qquad \delta\omega^{\,\mu\rho} + \delta\omega^{\,\rho\mu} = 0$$

and consequently

$$\begin{aligned}\Delta V_\mu(x) &= \delta\omega^{\rho\sigma}\, g_{\mu\rho}\, V_\sigma(x)\\ &\equiv -\tfrac{1}{2}\,\mathrm{i}\,\delta\omega^{\rho\sigma}\,(S_{\rho\sigma})_\mu^{\ \nu}\, V_\nu(x)\end{aligned}\tag{2.23}$$

the generators of the total variation of the covariant vector field under a Poincaré transformation being the *relativistic spin angular momentum* matrices (2.7), the action of which actually reads

$$\begin{aligned}(S_{\rho\sigma})_\mu^{\ \nu}\, V_\nu(x) &= \mathrm{i}\, g_{\mu\rho}\, V_\sigma(x) - \mathrm{i}\, g_{\sigma\mu}\, V_\rho(x)\\ &\equiv S_{\rho\sigma} * V_\mu(x)\end{aligned}\tag{2.24}$$

in which

$$(S_{\rho\sigma})_\mu^{\ \nu} = \mathrm{i}\, g_{\mu\rho}\,\delta_\sigma^\nu - \mathrm{i}\, g_{\mu\sigma}\,\delta_\rho^\nu \tag{2.25}$$

The above expression can be readily checked taking the four gradient of a real scalar field. As a matter of fact we obtain

$$\partial_\mu\phi(x) \equiv V_\mu(x) \tag{2.26}$$

so that from eq. (2.19) we find the infinitesimal transformation law

$$\begin{aligned}\Delta V_\mu(x) &= \Delta\,\partial_\mu\phi(x)\\ &= [\,\Delta\,,\,\partial_\mu\,]\,\phi(x) + \partial_\mu\Delta\,\phi(x)\\ &= [\,\Delta\,,\,\partial_\mu\,]\,\phi(x)\end{aligned}\tag{2.27}$$

Now we have

$$\begin{aligned}[\,\Delta\,,\,\partial_\mu\,] &= [\,\delta\,,\,\partial_\mu\,] + [\,\delta x\cdot\partial\,,\,\partial_\mu\,]\\ &= \delta\omega^{\lambda\nu}\, g_{\nu\rho}\,[\,x^\rho\,,\,\partial_\mu\,]\,\partial_\lambda\\ &= -\,\delta\omega^{\lambda\nu}\, g_{\nu\rho}\,\delta_\mu^\rho\,\partial_\lambda\\ &= \delta\omega^{\lambda\nu}\, g_{\lambda\mu}\,\partial_\nu\end{aligned}\tag{2.28}$$

and thereby

$$\Delta\,\partial_\mu\phi(x) = \delta\omega^{\rho\sigma}\, g_{\mu\rho}\,\partial_\sigma\phi(x)$$

whence eq. (2.23) immediately follows.

From the symbolic relation (2.15) we obtain the expression for the local variation of a relativistic covariant vector wave field

$$\begin{aligned}\delta V_\rho(x) &= \Delta V_\rho(x) - \delta x^\mu\,\partial_\mu V_\rho(x) = -\tfrac{1}{2}\,\mathrm{i}\,\delta\omega^{\mu\nu}\, S_{\mu\nu} * V_\rho(x)\\ &- \delta\omega^{\mu\nu}\, x_\nu\,\partial_\mu V_\rho(x) - \delta\omega^\mu\,\partial_\mu V_\rho(x)\\ &= -\tfrac{1}{2}\,\mathrm{i}\,\delta\omega^{\mu\nu}\, M_{\mu\nu} * V_\rho(x) + \mathrm{i}\,\delta\omega^\mu\, P_\mu\, V_\rho(x)\end{aligned}\tag{2.29}$$

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} = x_{\mu} P_{\nu} - x_{\nu} P_{\mu} + S_{\mu\nu}$$

Well–known examples of vector and tensor wave fields are the vector potential and the field strength of the electromagnetic field

$$\begin{aligned}
& A'_{\mu}(x') = \Lambda_{\mu}^{\ \nu} A_{\nu}(x) \\
& \Delta A_{\mu}(x) = \delta\omega^{\lambda\nu} g_{\mu\lambda} A_{\nu}(x) \\
& F'_{\mu\nu}(x') = \Lambda_{\mu}^{\ \rho} \Lambda_{\nu}^{\ \sigma} F_{\rho\sigma}(x) \\
& \Delta F_{\mu\nu}(x) = \delta\omega^{\lambda\rho} \left[g_{\mu\lambda} F_{\rho\nu}(x) + g_{\nu\lambda} F_{\mu\rho}(x) \right]
\end{aligned} \tag{2.30}$$

A tensor wave field of any rank with many contravariant and covariant indices will exploit a local variation in accordance to the straightforward generalization of the infinitesimal change (2.29). In particular, the action of the relativistic spin matrix on a tensor wave field will be the (algebraic) sum of expressions like (2.7), one for each index. For instance, the action of the spin matrix on the electromagnetic field strength is given by

$$\begin{aligned}
S_{\rho\sigma} * F_{\mu\nu}(x) &= \mathrm{i}\, g_{\rho\mu} F_{\sigma\nu}(x) - \mathrm{i}\, g_{\sigma\mu} F_{\rho\nu}(x) \\
&+ i\, g_{\rho\nu} F_{\mu\sigma}(x) - \mathrm{i}\, g_{\sigma\nu} F_{\mu\rho}(x)
\end{aligned} \tag{2.31}$$

If we consider a full parity transformation (1.20) $\Lambda_P \in L_-^{\uparrow}$ we have

$$A'_{\mu}(\Lambda_P x) = (\Lambda_P)_{\mu}^{\ \nu} A_{\nu}(x) \tag{2.32}$$

that yields

$$A'_0(x_0, -\mathbf{x}) = A_0(x_0, \mathbf{x}) \qquad \mathbf{A}'(x_0, -\mathbf{x}) = -\,\mathbf{A}(x_0, \mathbf{x}) \tag{2.33}$$

Conversely, a covariant *pseudovector* wave field will be defined by the transformation law

$$\widetilde{V}'_{\mu}(\Lambda x) = (\det\Lambda)\, \Lambda_{\mu}^{\ \nu} \widetilde{V}_{\nu}(x) \qquad \forall\, \Lambda \in L_-^{\uparrow} \tag{2.34}$$

so that under parity

$$\widetilde{V}'_0(x_0, -\mathbf{x}) = -\,\widetilde{V}_0(x_0, \mathbf{x}) \qquad \widetilde{\mathbf{V}}'(x_0, -\mathbf{x}) = \widetilde{\mathbf{V}}(x_0, \mathbf{x}) \tag{2.35}$$

2.1.3 The Spinor Fields

The two irreducible fundamental representations $\boldsymbol{\tau}_{\frac12 0}$ and $\boldsymbol{\tau}_{0\frac12}$ of the homogeneous Lorentz group can be realized by means of $SL(2,\mathbb{C})$, *i.e.* the group of complex 2×2 matrices of unit determinant. The $SL(2,\mathbb{C})$ matrices belonging to $\boldsymbol{\tau}_{\frac12 0}$ act upon the so called *left Weyl's two–component spinors*, whilst the $SL(2,\mathbb{C})$ matrices belonging to $\boldsymbol{\tau}_{0\frac12}$ act upon the so called *right Weyl's two–component spinors*.

In any neighbourhood of the unit element, the $SL(2,\mathbb{C})$ matrices can always be presented in the exponential form

$$\Lambda_L \equiv \exp\left\{\tfrac12\,\mathrm{i}\,\sigma_k\,(\alpha_k - \mathrm{i}\,\eta_k)\right\} \tag{2.36}$$

$$\Lambda_R \equiv \exp\left\{\tfrac12\,\mathrm{i}\,\sigma_k\,(\alpha_k + \mathrm{i}\,\eta_k)\right\} \tag{2.37}$$

where $\alpha_k, \beta_k, \eta_k = \mathrm{Arsh}\left(\beta_k(1-\beta_k^2)^{-1/2}\right)$ $(k=1,2,3)$ are respectively the angle, velocity (1.21) and rapidity (1.22) parameters of the Lorentz group, whereas σ_k $(k=1,2,3)$ are the Pauli matrices. Notice that

$$\sigma_2\,\sigma_k\,\sigma_2 = -\,\sigma_k^* = -\,\sigma_2^{\top} \qquad (k=1,2,3) \tag{2.38}$$

$$\begin{aligned} &\sigma_j\,\sigma_k = \delta_{jk} + \mathrm{i}\,\varepsilon_{jkl}\,\sigma_l \qquad &&\{\sigma_j\,\sigma_k\} = 2\delta_{jk} \\ &[\,\sigma_j\,\sigma_k\,] = 2\mathrm{i}\,\varepsilon_{jkl}\,\sigma_l &&(\,j,k,l = 1,2,3\,) \end{aligned} \tag{2.39}$$

It is clear that for $\beta_k = \eta_k = 0$ $(k=1,2,3)$, *i.e.* for rotations, we have that $\Lambda_L(\alpha) = \Lambda_R(\alpha) \in SU(2)$ whereas for $\alpha_k = 0$ $(k=1,2,3)$, *i.e.* for boosts, the matrices $\Lambda_{L\,,\,R}(\eta)$ are hermitean and nonunitary.

Let us therefore introduce the relativistic two–component Weyl's spinor wave fields

$$\psi_L(x) \equiv \begin{pmatrix} \psi_{L\,1}(x) \\ \psi_{L\,2}(x) \end{pmatrix} \qquad \psi_R(x) \equiv \begin{pmatrix} \psi_{R\,1}(x) \\ \psi_{R\,2}(x) \end{pmatrix} \tag{2.40}$$

which, by definition, transform according to

$$\psi_L'(x') = \Lambda_L\,\psi_L(x) \qquad \psi_R'(x') = \Lambda_R\,\psi_R(x) \tag{2.41}$$

The infinitesimal form of the above transformation laws give rise to the total variations

$$\begin{aligned} \Delta\,\psi_L(x) &= \tfrac12\,\mathrm{i}\,(\sigma_j\,\delta\alpha_j - \mathrm{i}\,\sigma_k\,\delta\beta_k)\,\psi_L(x) \\ &= -\,\tfrac12\,\mathrm{i}\left(\sigma_j\,\tfrac12\,\varepsilon_{jkl}\,\delta\,\omega^{kl} + \mathrm{i}\,\sigma_k\,\delta\,\omega^{0k}\right)\psi_L(x) \\ &\equiv -\,\tfrac12\,\mathrm{i}\,\delta\,\omega^{\rho\sigma}\,(S_{\rho\sigma})_L\,\psi_L(x) \end{aligned} \tag{2.42}$$

and quite analogously

$$
\begin{aligned}
\Delta\,\psi_R\,(x) &= \tfrac{1}{2}\,\mathrm{i}\,(\sigma_j\,\delta\alpha_{\,j}+\mathrm{i}\,\sigma_k\,\delta\beta_k)\,\psi_R\,(x)\\
&= -\,\tfrac{1}{2}\,\mathrm{i}\left(\sigma_j\,\tfrac{1}{2}\,\varepsilon_{jk\ell}\,\delta\,\omega^{\,k\ell}-\mathrm{i}\,\sigma_k\,\delta\,\omega^{\,0k}\right)\psi_R\,(x)\\
&\equiv -\,\tfrac{1}{2}\,\mathrm{i}\,\delta\,\omega^{\,\rho\sigma}\,(S_{\rho\sigma})_R\,\psi_R\,(x)
\end{aligned}
\tag{2.43}
$$

whence we identify

$$
(S_{k\ell})_L \;=\; \tfrac{1}{2}\,\varepsilon_{jk\ell}\,\sigma_j \;=\; (S_{k\ell})_R \tag{2.44}
$$

$$
(S_{0k})_L \;=\; \tfrac{1}{2}\,\mathrm{i}\,\sigma_k \;=\; -\,(S_{0k})_R \tag{2.45}
$$

From the symbolic relation (2.15) we can easily obtain the expression for the local variation of both relativistic Weyl's spinor fields.

The $SL(2,\mathbb{C})$ matrices do satisfy some important properties :

(a) $\Lambda_L^{-1}=\Lambda_R^{\dagger} \qquad \Lambda_R^{-1}=\Lambda_L^{\dagger}$

(b) $\sigma_2\,\Lambda_L\,\sigma_2=\Lambda_R^{*} \qquad \sigma_2\,\Lambda_R\,\sigma_2=\Lambda_L^{*}$

(c) $\Lambda_L^{\top}=\sigma_2\,\Lambda_L^{-1}\,\sigma_2 \qquad \Lambda_R^{\top}=\sigma_2\,\Lambda_R^{-1}\,\sigma_2$

`Proof`

(a) We have

$$
\begin{aligned}
\Lambda_R^{\dagger} &= \exp\left\{-\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k-\mathrm{i}\,\eta_k)\right\}\\
&= \exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k\,(-\,\alpha_k+\mathrm{i}\,\eta_k)\right\}\\
&= \Lambda_L(-\,\alpha,-\,\eta)=\Lambda_L^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_L^{\dagger} &= \exp\left\{-\,\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k+\mathrm{i}\,\eta_k)\right\}\\
&= \exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k\,(-\,\alpha_k-\mathrm{i}\,\eta_k)\right\}\\
&= \Lambda_R(-\,\alpha,-\,\eta)=\Lambda_R^{-1}
\end{aligned}
$$

(b) From the definition of the exponential of a matrix we get

$$
\sigma_2\,\Lambda_L\,\sigma_2 \;=\; \sigma_2\,\exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k-\mathrm{i}\,\eta_k)\right\}\sigma_2
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^{n}\frac{1}{n!}\,\sigma_2\,\sigma_{k_1}\,\sigma_{k_2}\,\cdots\,\sigma_{k_n}\,\sigma_2 \\
&\times\ (\alpha_{k_1}-\mathrm{i}\,\eta_{k_1})\,(\alpha_{k_2}-\mathrm{i}\,\eta_{k_2})\,\cdots\,(\alpha_{k_n}-\mathrm{i}\,\eta_{k_n}) \\
&= \sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^{n}\frac{1}{n!}\,\sigma_{k_1}^{*}\,\sigma_{k_2}^{*}\,\cdots\,\sigma_{k_n}^{*} \\
&\times\ (-\,\alpha_{k_1}+\mathrm{i}\,\eta_{k_1})\,(-\,\alpha_{k_2}+\mathrm{i}\,\eta_{k_2})\,\cdots\,(-\,\alpha_{k_n}+\mathrm{i}\,\eta_{k_n}) \\
&= \exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k^{*}\,(-\,\alpha_k+\mathrm{i}\,\eta_k)\right\} \\
&= \left(\exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k+\mathrm{i}\,\eta_k)\right\}\right)^{*} = \Lambda_R^{*}
\end{aligned}
$$

and in the very same way we prove that $\sigma_2\,\Lambda_R\,\sigma_2 = \Lambda_L^{*}$.

(c) Finally we get

$$
\begin{aligned}
\sigma_2\,\Lambda_L^{-1}\,\sigma_2 &= \sigma_2\,\Lambda_R^{\dagger}\,\sigma_2 \\
&= \sigma_2\,\exp\left\{-\,\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k-\mathrm{i}\,\eta_k)\right\}\,\sigma_2 \\
&= \left(\exp\left\{\frac{\mathrm{i}}{2}\,\sigma_k\,(\alpha_k-\mathrm{i}\,\eta_k)\right\}\right)^{\top} = \Lambda_L^{\top}
\end{aligned}
$$

and repeating step–by–step we prove that $\sigma_2\,\Lambda_R^{-1}\,\sigma_2 = \Lambda_R^{\top}$ *q.e.d.*

The above listed relations turn out to be rather useful to single out Lorentz invariant combinations out of the Weyl's spinors. Consider for instance

$$
\left(\sigma_2\,\psi_L^{*}\right)' = \sigma_2\,(\Lambda_L\psi_L)^{*} = \sigma_2\Lambda_L^{*}\sigma_2\,\sigma_2\psi_L^{*} = \Lambda_R\,\sigma_2\psi_L^{*}
$$

which means that $\sigma_2\,\psi_L^{*}\in\boldsymbol{\tau}_{0\,\frac12}$ and correspondingly $\sigma_2\,\psi_R^{*}\in\boldsymbol{\tau}_{\frac12\,0}$.

We show now that the antisymmetric combination of the two Weyl representations of the Lorentz group transforms according to the scalar representation. As a matter of fact, on the one hand we find

$$
\chi_L^{\top}\,\sigma_2\,\psi_L = -\mathrm{i}\,(\chi_{L\,1}\psi_{L\,2}-\chi_{L\,2}\psi_{L\,1}) \tag{2.46}
$$

On the other hand we have

$$
(\chi_L^{\top}\,\sigma_2\,\psi_L)' = \chi_L^{\top}\Lambda_L^{\top}\,\sigma_2\,\Lambda_L\psi_L = \chi_L^{\top}\,\sigma_2\,\psi_L\in\boldsymbol{\tau}_{0\,0} \tag{2.47}
$$

From the antisymmetry of the above invariant combination, it follows that $\psi_L^{\top}\,\sigma_2\,\psi_L\equiv 0$ for ordinary c–number valued Weyl's left spinors.

If we now take $\chi_L \equiv \sigma_2\,\psi_R^*$ then we obtain the complex invariants

$$\mathfrak{I} = \psi_R^{\top *}\sigma_2^{\top}\sigma_2\,\psi_L = -\,\psi_R^{\dagger}\,\psi_L \qquad \mathfrak{I}^* = -\,\psi_L^{\dagger}\,\psi_R \tag{2.48}$$

We can build up a four vector out of a single Weyl left spinor fields. To this purpose, let me recall the transformation laws of a contravariant four vector under passive infinitesimal boosts and rotations respectively

$$\delta V^0 = \varepsilon^0{}_k\,V^k = -\,\varepsilon^{0k}\,V^k = -\,\delta\beta_k\,V^k \tag{2.49}$$
$$\delta V^j = \varepsilon^j{}_0\,V^0 = \varepsilon^{j0}\,V^0 = -\,\delta\beta_j\,V^0 \tag{2.50}$$
$$\delta V^j = \varepsilon^j{}_k\,V^k = -\,\varepsilon^{jk}\,V^k = -\,\varepsilon^{jk\ell}\,V^k\,\delta\alpha_\ell \tag{2.51}$$

Consider in fact the left combination

$$\psi_L^\dagger(x)\,\sigma^\mu\,\psi_L(x) \qquad \sigma^\mu \;\equiv\; (\mathbf{1}, -\,\sigma_k) \qquad (\,k = 1,2,3\,)$$

Under a passive infinitesimal Lorentz transformation the total variation (2.42) yields

$$\begin{aligned}
&\Delta\Big(\psi_L^\dagger(x)\,\sigma^\mu\,\psi_L(x)\Big) \;=\\
-&\;\tfrac12\,\mathrm{i}\,\psi_L^\dagger(x)\,\sigma_k\,\sigma^\mu\,\psi_L(x)\,(\delta\alpha_k + \mathrm{i}\,\delta\beta_k)\\
+&\;\tfrac12\,\mathrm{i}\,\psi_L^\dagger(x)\,\sigma^\mu\,\sigma_k\,\psi_L(x)\,(\delta\alpha_k - \mathrm{i}\,\delta\beta_k)\\
=&\;\tfrac12\,\mathrm{i}\,\psi_L^\dagger(x)\,[\,\sigma^\mu\,,\,\sigma_k\,]\,\psi_L(x)\,\delta\alpha_k + \tfrac12\,\psi_L^\dagger(x)\,\{\sigma_k\,,\,\sigma^\mu\}\,\psi_L(x)\,\delta\beta_k\\
=&\;\begin{cases} \psi_L^\dagger(x)\,\sigma_k\,\psi_L(x)\,\delta\beta_k & (\mu = 0)\\ -\,\psi_L^\dagger(x)\,\psi_L(x)\,\delta\beta_j + \varepsilon_{jk\ell}\,\psi_L^\dagger(x)\,\sigma_\ell\,\psi_L(x)\,\delta\alpha_k & (\mu = j)\end{cases}\\
=&\;-\,\tfrac12\,\mathrm{i}\,\delta\omega^{\rho\sigma}\,(S_{\rho\sigma})^\mu{}_\nu\,\psi_L^\dagger(x)\,\sigma^\nu\,\psi_L(x)
\end{aligned}\tag{2.52}$$

which means that the left-handed combination

$$V_L^\mu(x) \equiv \psi_L^\dagger(x)\,\sigma^\mu\,\psi_L(x) = V_L^{\mu\,\dagger}(x) \qquad \sigma^\mu = (\mathbf{1}, -\,\sigma_k)$$

transforms under the Lorentz group like a contravariant real vector field, that is

$$V_L^{\prime\mu}(x') = \Lambda^\mu{}_\nu\,\psi_L^\dagger(x)\,\sigma^\nu\,\psi_L(x) = \Lambda^\mu{}_\nu\,V_L^\nu(x) \tag{2.53}$$

In the very same way one can see that the right combination

$$V_R^\mu(x) \equiv \psi_R^\dagger(x)\,\bar\sigma^\mu\,\psi_R(x) = V_R^{\mu\,\dagger}(x) \qquad \bar\sigma^\mu = (\mathbf{1}, \sigma_k)$$

transform under the Lorentz group like a contravariant real vector field

$$V_R^{\prime\mu}(x') = \Lambda^\mu{}_\nu\,\psi_R^\dagger(x)\,\bar\sigma^\nu\,\psi_R(x) = \Lambda^\mu{}_\nu\,V_R^\nu(x) \tag{2.54}$$

In conclusion we see that the following relationships hold true : namely,

$$\Lambda_L^\dagger\,\sigma^\mu\,\Lambda_L = \Lambda^\mu{}_\nu\,\sigma^\nu \qquad \Lambda_R^\dagger\,\bar\sigma^\mu\,\Lambda_R = \Lambda^\mu{}_\nu\,\bar\sigma^\nu \tag{2.55}$$

It becomes now easy to build up the Lorentz invariant real kinetic terms

$$\begin{aligned} \mathfrak{T}_L &= \tfrac{1}{2}\,\psi_L^\dagger(x)\,\sigma^\mu\,\mathrm{i}\,\partial_\mu\psi_L(x) - \tfrac{1}{2}\,\mathrm{i}\,\partial_\mu\psi_L^\dagger(x)\,\sigma^\mu\,\psi_L(x) \\ &\equiv \tfrac{1}{2}\,\psi_L^\dagger(x)\,\sigma^\mu\,\mathrm{i}\,\overleftrightarrow{\partial}_\mu\psi_L(x) \qquad (2.56) \\ \mathfrak{T}_R &= \tfrac{1}{2}\,\psi_R^\dagger(x)\,\bar\sigma^\mu\,\mathrm{i}\,\partial_\mu\psi_R(x) - \tfrac{1}{2}\,\mathrm{i}\,\partial_\mu\psi_R^\dagger(x)\,\bar\sigma^\mu\,\psi_R(x) \\ &\equiv \tfrac{1}{2}\,\psi_R^\dagger(x)\,\bar\sigma^\mu\,\mathrm{i}\,\overleftrightarrow{\partial}_\mu\psi_R(x) \qquad (2.57) \end{aligned}$$

When the full Lorentz group is a concern, we have already seen (1.24) that it is necessary to consider the direct sum of the two inequivalent irreducible Weyl's representation. In so doing, we are led to the so called four components bispinor or Dirac relativistic spinor wave field

$$\psi(x) \equiv \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \psi_{L\,1}(x) \\ \psi_{L\,2}(x) \\ \psi_{R\,1}(x) \\ \psi_{R\,2}(x) \end{pmatrix} \tag{2.58}$$

The full parity transformation, also named space inversion,

$$\mathbb{P}\;:\;\psi_{L\,,\,R} \leftrightarrow \psi_{R\,,\,L}$$

is then represented by the 4×4 matrix

$$\gamma^0 \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

so that

$$\psi'(x') = \psi'(x^0, -\mathbf{x}) = (\mathbb{P}\,\psi)(x) = \gamma^0\,\psi(x) = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} \tag{2.59}$$

The left and right Weyl's components can be singled out by means of the two projectors

$$P_L \equiv \tfrac{1}{2}\,(\mathbf{1} - \gamma_5) \qquad P_R \equiv \tfrac{1}{2}\,(\mathbf{1} + \gamma_5)$$

where

$$\gamma_5 = \gamma^5 \equiv \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Starting from the Weyl's invariants (2.48) one can easily build up the parity-even Dirac invariant

$$\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R = \psi^\dagger \gamma^0 \psi \equiv \bar{\psi}\psi \tag{2.60}$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$ is said to be the *adjoint spinor*. In the very same way we can construct the parity even and Lorentz invariant Dirac kinetic term

$$\mathfrak{T}_D = \tfrac{1}{2}\,\bar{\psi}(x)\,\gamma^\mu\,\mathrm{i}\,\overleftrightarrow{\partial}_\mu \psi(x) \tag{2.61}$$

which is real, where we have eventually introduced the set of matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.62}$$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.63}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{2.64}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.65}$$

$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.66}$$

The above set of five 4×4 matrices are said to be the *Dirac matrices in the Weyl or chiral or even spinorial representation.* The gamma matrices do satisfy the so called *Clifford algebra*

$$\{\gamma^{\mu}\,,\,\gamma^{\nu}\} = 2g^{\mu\nu} \qquad \{\gamma^{\mu}\,,\,\gamma_5\} = 0 \tag{2.67}$$

in which

$$\gamma_5 \equiv \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3$$

Notice that we have the hermitean conjugation properties

$$\gamma^{\mu\,\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0} \qquad \gamma_5^{\dagger} = \gamma_5 \tag{2.68}$$

and moreover

$$\gamma^{\mu} = \left(\begin{array}{cc} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{array}\right) \qquad \bar{\sigma}^{\mu} \equiv (\mathbf{1}, \boldsymbol{\sigma}) \qquad \sigma^{\mu} \equiv (\mathbf{1}, -\,\boldsymbol{\sigma})$$

$$\gamma^{0}\gamma^{\mu} = \left(\begin{array}{cc} \sigma^{\mu} & 0 \\ 0 & \bar{\sigma}^{\mu} \end{array}\right) \equiv \alpha^{\mu}$$

We have already seen the transformation laws of the left–handed (2.42) and right–handed (2.43) Weyl's spinors under the restricted Lorentz group. If we consider an infinitesimal boost $\delta\,\omega^{\,kl} = 0$ $(k, l = 1, 2, 3)$ both transformation laws can be written in terms of a Dirac spinor as

$$\Delta\,\psi(x) = \tfrac{1}{4}\,\delta\,\omega_{\,0k}\,(\gamma^0\gamma^k - \gamma^k\gamma^0)\,\psi(x) \qquad (k = 1, 2, 3) \tag{2.69}$$

Analogously, the transformation laws under an infinitesimal rotation $\delta\,\omega^{\,0k} = 0$ $(k = 1, 2, 3)$ can be written together in the form

$$\Delta\,\psi(x) = \tfrac{1}{8}\,\delta\,\omega^{\,jk}(\gamma^j\gamma^k - \gamma^k\gamma^j)\,\psi(x) \qquad (\,j, k = 1, 2, 3) \tag{2.70}$$

This means that, if we introduce

$$(S^{\mu\nu})_D \;\equiv\; \sigma^{\mu\nu} \;=\; \frac{\mathrm{i}}{4}\,[\,\gamma^{\mu}\,,\,\gamma^{\nu}\,] \tag{2.71}$$

we eventually obtain that the transformation law for the Dirac spinors under the restricted Lorentz group becomes

$$\Delta\,\psi(x) \;=\; -\,\frac{\mathrm{i}}{2}\,\sigma^{\mu\nu}\,\psi(x)\,\delta\,\omega_{\,\mu\nu} \tag{2.72}$$

Hence the spin angular momentum tensor operator for the relativistic Dirac spinor wave field reads

$$\sigma_{\,\mu\nu} \;=\; g_{\,\mu\rho}\,g_{\,\nu\sigma}\,\frac{1}{4\mathrm{i}}\,[\,\gamma^{\rho}\,,\,\gamma^{\sigma}\,] \tag{2.73}$$

the spatial components being hermitean whilst the spatial temporal components being antihermitean, that is

$$(\,\sigma^{\mu\nu}\,)^{\dagger} = \gamma^{0}\,\sigma^{\mu\nu}\,\gamma^{0} \qquad\qquad \sigma_{\mu\nu} = \gamma^{0}\,\sigma^{\dagger}_{\mu\nu}\,\gamma^{0} \tag{2.74}$$

By the very construction the six components of the spin angular momentum tensor of the Dirac field enjoy the Lie algebra of the Lorentz group

$$[\,\sigma_{\mu\nu}\,,\,\sigma_{\rho\sigma}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,\sigma_{\nu\sigma}\;+\;\text{cyclic permutations}$$

It is worthwhile to remark that the above construction keeps true in any D-dimensional space with a symmetry group $O(m,n)$ in which

$$D = m + n \qquad m \geq 0 \quad n \geq 0$$

$$x^{2} = \sum_{k=1}^{m} x_{k}^{2} - \sum_{j=1}^{n} x_{m+j}^{2}$$

For any finite passive transformation of the restricted Lorentz group we have

$$\psi^{\prime}(x^{\prime}) \;=\; \Lambda_{\frac{1}{2}}\,(\omega)\,\psi\,(x) \;=\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\sigma_{\mu\nu}\,\omega^{\mu\nu}\right\}\,\psi\,(x) \tag{2.75}$$

As an example, for a boost along the positive OY axis we find

$$\begin{aligned}
\Lambda_{\frac{1}{2}}\,(\eta) &= \exp\left\{-\,\mathrm{i}\,\sigma_{02}\,\omega^{02}\right\} = \cosh\frac{\eta}{2} - \gamma^{0}\gamma^{2}\sinh\frac{\eta}{2} = \Lambda^{\dagger}_{\frac{1}{2}}(\eta) \\
&= \begin{pmatrix}
\cosh\eta/2 & \mathrm{i}\sinh\eta/2 & 0 & 0 \\
-\,\mathrm{i}\sinh\eta/2 & \cosh\eta/2 & 0 & 0 \\
0 & 0 & \cosh\eta/2 & -\,\mathrm{i}\sinh\eta/2 \\
0 & 0 & \mathrm{i}\sinh\eta/2 & \cosh\eta/2
\end{pmatrix}
\end{aligned}$$

where $\sinh\eta = v_{y}\,(1 - v_{y}^{2})^{-1/2}\;(v_{y} > 0)\,.$

From the hermitean conjugation property (2.74) we can write

$$\begin{aligned}
\Lambda^{\dagger}_{\frac{1}{2}}\,(\omega) &= \exp\left\{\tfrac{1}{2}\,\mathrm{i}\,\sigma^{\dagger}_{\mu\nu}\,\omega^{\mu\nu}\right\} \\
&= \sum_{n=0}^{\infty}\frac{1}{n\,!}\prod_{\jmath=1}^{n}\left(\frac{\mathrm{i}}{2}\right)\,\gamma^{0}\,\sigma_{\mu_{\jmath}\rho_{\jmath}}\,\gamma^{0}\,\omega^{\mu_{\jmath}\rho_{\jmath}} \\
&= \gamma^{0}\sum_{n=0}^{\infty}\frac{1}{n\,!}\left(\frac{\mathrm{i}}{2}\right)^{n}\,(\,\sigma_{\mu\nu}\,\omega^{\mu\nu}\,)^{n}\,\gamma^{0} \\
&= \gamma^{0}\,\exp\left\{\tfrac{1}{2}\,\mathrm{i}\,\sigma_{\mu\nu}\,\omega^{\mu\nu}\right\}\,\gamma^{0} \\
&= \gamma^{0}\,\Lambda^{-1}_{\frac{1}{2}}\,(\omega)\,\gamma^{0} = \gamma^{0}\,\Lambda_{\frac{1}{2}}\,(-\,\omega)\,\gamma^{0}
\end{aligned} \tag{2.76}$$

which entails in turn the two further relations

$$\gamma^0\,\Lambda^{\dagger}_{\frac{1}{2}}(\omega)\,\gamma^0=\Lambda^{-1}_{\frac{1}{2}}(\omega)=\Lambda_{\frac{1}{2}}(-\,\omega) \tag{2.77}$$

$$\Big(\Lambda^{-1}_{\frac{1}{2}}(\omega)\Big)^{\dagger}\,\gamma^0\,\Lambda^{-1}_{\frac{1}{2}}(\omega)=\Lambda^{\dagger}_{\frac{1}{2}}(-\omega)\,\gamma^0\,\Lambda^{-1}_{\frac{1}{2}}(\omega)=\gamma^0 \tag{2.78}$$

From the Lorentz invariance of the Dirac real kinetic term (2.61) it follows that the bilinear

$$\bar\psi(x)\,\gamma^{\mu}\,\psi(x)\;\equiv\;J^{\mu}(x)$$

transforms as a contravariant four vector and is thereby named the *Dirac vector current*. Hence, by making use of equation (2.77), the finite transformation law immediately follows, *viz.,*

$$\Lambda^{-1}_{\frac{1}{2}}(\omega)\,\gamma^{\lambda}\,\Lambda_{\frac{1}{2}}(\omega)=\Lambda^{\lambda}{}_{\kappa}\,\gamma^{\kappa} \tag{2.79}$$

$$\gamma^{\lambda}=\Lambda^{\lambda}{}_{\kappa}\,\Lambda_{\frac{1}{2}}(\omega)\,\gamma^{\kappa}\,\Lambda^{-1}_{\frac{1}{2}}(\omega) \tag{2.80}$$

As we have seen above, it turns out that $\sigma_2\psi_L^*\in\boldsymbol{\tau}_{0\,\frac{1}{2}}$ while $\sigma_2\psi_R^*\in\boldsymbol{\tau}_{\frac{1}{2}\,0}$. Thus, we can build up the *charge conjugated spinor* of a given relativistic Dirac wave field $\psi(x)$ as follows :

$$\psi\quad\mapsto\quad\psi^{\,c}=C\psi^*=\begin{pmatrix}\sigma_2\,\psi_R^*\\-\,\sigma_2\,\psi_L^*\end{pmatrix}\qquad C=\gamma^2 \tag{2.81}$$

Notice that $(\psi^{\,c})^{\,c}=\gamma^2\,(\gamma^2\,\psi^*)^*=\psi$. As a consequence, starting from a solely left-handed or right-handed Weyl spinor, it is possible to build up *charge self-conjugated Majorana bispinors*

$$\begin{aligned}\psi_L^M&=\begin{pmatrix}\psi_L\\-\,\sigma_2\,\psi_L^*\end{pmatrix}=(\psi_L^M)^{\,c}\\ \psi_R^M&=\begin{pmatrix}\sigma_2\,\psi_R^*\\\psi_R\end{pmatrix}=(\psi_R^M)^{\,c}\end{aligned} \tag{2.82}$$

The Majorana charge self-conjugated spinors are Weyl spinors in a four components form and thereby their field degrees of freedom content is half of that one of a Dirac spinor wave field.

2.2 The Action Principle

In the previous sections we have seen how to build up Poincaré invariant expressions out of the classical relativistic wave fields corresponding to the irreducible tensor and spinor representations of the inhomogeneous Lorentz group. The requirement of Poincaré invariance will ensure that these classical field theories will obey the axioms of the special theory of relativity.

The general properties that will specify the action for the collection of the classical relativistic wave field functions $u_A(x)$ $(A = 1, 2, \ldots, N)$ will be assumed in close analogy with the paradigmatic case of the electromagnetic field.

1. The action integrand $\mathcal{L}(x)$ is called the Lagrange density or lagrangian for short : in the absence of external pre–assigned background fields, the lagrangian can not explicitely depend on the coordinates x^μ , so as to ensure spacetime translation invariance, and must be a Lorentz invariant to ensure that the corresponding theory will obey the axioms of special relativity.

2. To fulfill causality, the differential equations for the field functions must be at most of the second order in time, in such a way that the related Cauchy problem has a unique solution. Classical field theories described by differential equations of order higher than the second in time will typically develop non–causal solutions, a well–known example being the Abraham–Lorentz equation [1] of electrodynamics, which is a third order in time differential equation that encodes the effects of the radiation reaction and shows acausal effects such as pre–acceleration of charged particles yet to be hit by radiation.

3. The wave equations for all the fundamental fields that describe matter and radiation are assumed to be partial differential equations and not integro–differential equations, which do satisfy Lorentz covariance in accordance with the special theory of relativity : as a consequence the lagrangian must be a Lorentz invariant local functional of the field functions and their first partial derivatives

   $$\mathcal{L}(x) = \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)$$

   In the classical theory the action has the physical dimensions of an angular momentum, *i.e.* g cm^2 s^{-1} , or equivalently of the Planck's

[1] M. Abraham, Ann. Phys. **10**, 105 (1903), H. A. Lorentz, Amst. Versl. **12**, 986 (1904); *cfr.* [1] pp. 578–610.

constant. In the natural unit system the action is dimensionless and the lagrangian in four spacetime dimensions has natural dimensions of cm^{-4}.

4. According to the variational principle of classical mechanics, the action must be real and must exhibit a local minimum in correspondence of the Euler–Lagrange equations of motion : in classical physics complex potentials lead to absorption, *i.e.* disappearance of matter into nothing, a phenomenon that will not be considered in the sequel – it will be found *a posteriori* that a real classical action is crucial to obtain a satisfactory quantum field theory where the total probability is conserved.

5. The astonishing phenomenological and theoretical success of the gauge theories in the construction of the present day standard model of the fundamental interactions in particle physics does indeed suggest that the action functional will be invariant under further symmetry groups of transformations beyond the inhomogeneous Lorentz group. Such a kind of transformations does not act upon the spacetime coordinates and will be thereby called *internal symmetry groups of transformations.* These transformations will involve new peculiar field degrees of freedom such as the electric charge, the weak charge, the colour charge and maybe other charges yet to be discovered. In particular, gauge theories are described by action functionals wich are invariant under local – *i.e.* spacetime point dependent – transformations among those internal degrees of freedom.

Consider the action

$$S(t_i, t_f\,;[\,u\,]) = \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \mathcal{L}\Big(u(x), \partial_\mu u(x)\Big)$$

where we shall denote by

$$u(x) = \{u_A(x) \mid A = 1,2,\ldots,N\,,\ x = (x_0, \mathbf{x})\,,\ t_i \leq x_0 \leq t_f\,,\ \mathbf{x} \in \mathbb{R}^3\}$$

the collection of all classical relativistic local wave fields. The index $A = 1,2,\ldots,N < \infty$ runs over the Lorentz group as well as all the internal symmetry group representations, so that we can suppose the local wave field component to be real valued functions.

We recall that, by virtue of the principle of the least action, the field variations are assumed to be local and infinitesimal

$$u(x) \ \mapsto\ u'(x) = u(x) + \delta u(x) \qquad |\,\delta u(x)\,| \ll |\,u(x)\,|$$

and to vanish at the initial and final times t_i and t_f

$$\delta u(t_i\,,\mathbf{x}) = \delta u(t_f\,,\mathbf{x}) = 0 \qquad \forall\,\mathbf{x}\in\mathbb{R}^3 \tag{2.83}$$

A local variation with respect to the wave field amplitudes gives

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f}\mathrm{d}t\int\mathrm{d}\mathbf{x}\;\delta\,\mathcal{L}\Big(u(x),\partial_\mu u(x)\Big)\\
&= \int_{t_i}^{t_f}\mathrm{d}t\int\mathrm{d}\mathbf{x}\;\left[\frac{\delta\mathcal{L}}{\delta u(x)}\,\delta u(x)+\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\,\delta\,\partial_\mu u(x)\right]
\end{aligned}$$

The local infinitesimal variations do satisfy by definition

$$\delta\,\partial_\mu u(x) = \partial_\mu\,\delta u(x) \qquad \Rightarrow \qquad [\,\delta\,,\,\partial_\mu\,] = 0$$

so that

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f}\mathrm{d}x_0\int\mathrm{d}\mathbf{x}\;\left[\frac{\delta\mathcal{L}}{\delta u(x)}\;-\;\partial_\mu\,\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\right]\delta u(x)\\
&+\;\int_{t_i}^{t_f}\mathrm{d}x_0\int\mathrm{d}\mathbf{x}\;\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\;\delta u(x)\right]
\end{aligned}$$

The very last term can be rewritten, using the Gauß theorem, in the form

$$\begin{aligned}
&\int\mathrm{d}\mathbf{x}\left[\frac{\delta\mathcal{L}}{\delta\,\partial_0 u(x)}\;\delta u(x)\right]_{t_i}^{t_f}+\int_{t_i}^{t_f}\mathrm{d}x_0\int\mathrm{d}\mathbf{x}\;\boldsymbol{\nabla}\cdot\left[\frac{\delta\mathcal{L}}{\delta\,\boldsymbol{\nabla}u(x)}\;\delta u(x)\right]\\
&= \lim_{R\to\infty}\;R^2\int_{t_i}^{t_f}\mathrm{d}x_0\int\mathrm{d}\Omega\;\widehat{\mathbf{r}}\cdot\left[\frac{\delta\mathcal{L}}{\delta\,\boldsymbol{\nabla}u(x_0,R,\Omega)}\;\delta u(x_0,R,\Omega)\right]
\end{aligned}$$

where R is the ray of a very large sphere centered at $\mathbf{x}=0\,,\ \Omega=(\theta,\phi)$ is the solid angle in the three dimensional space and $\widehat{\mathbf{r}}$ is the radial unit vector, *i.e.* the exterior unit normal vector to the sphere. If we assume the asymptotic radial behaviour

$$\lim_{R\to\infty}\;R^2\,[\,\delta\mathcal{L}/\delta\,\partial_r u(x_0,R,\Omega)\,]\,\delta u(x_0,R,\Omega) = 0 \qquad \forall x_0\in[\,t_i,t_f\,] \tag{2.84}$$

where $\partial_r\equiv\widehat{\mathbf{r}}\cdot\boldsymbol{\nabla}$ is the radial derivative, then the above boundary term indeed disappears and consequently, from the arbitrariness of the local variations $\delta u(x)$, we eventually come to the Euler–Lagrange equations of motion for the classical relativistic wave field

$$\partial_\mu\,\frac{\delta\mathcal{L}}{\delta\partial_\mu u(x)}\;-\;\frac{\delta\mathcal{L}}{\delta u(x)}\;=\;0 \tag{2.85}$$

2.3 The Noether Theorem

For the construction of the constants of motion in field theory we shall use *Noether theorem* :

Amalie Emmy Noether
Erlangen 23.03.1882 – Brynn 14.04.1935
Invariante Varlationsprobleme, Nachr. d. König. Gesellsch. d. Wiss. Göttingen, Math–phys. Klasse (1918), 235–257
English translation M. A. Travel
Transport Theory and Statistical Physics **1** (1971), 183–207 .

This theorem states that to every continuous transformation of coordinates and fields which makes the variation of the action equal to zero there always corresponds a definite constant of motion, *i.e.* a combination of the field functions and their derivatives which remains conserved in time. Such a transformation of coordinates and fields will be called a *continuous symmetry* and will correspond to some representation of a Lie group of transformations of finite dimensions n .

In order to prove Noether theorem we shall consider the infinitesimal transformation of coordinates

$$x^{\mu} \quad \mapsto \quad x^{\prime\,\mu} = x^{\mu} + \delta x^{\mu} \qquad \delta x^{\mu} = X_{a}^{\,\mu}\,\delta\,\omega^{\,a} \tag{2.86}$$

with coefficients X_a^{μ} that may depend or not upon the spacetime points and s *spacetime independent* infinitesimal parameters

$$\delta\,\omega^{\,a} \qquad (a = 1, 2, \ldots, s)$$

which, for example, in the case of a transformation from the inhomogeneous Lorentz group include the infinitesimal translations $\delta\,\omega^{\,\mu}$ and homogeneous Lorentz transformations $\delta\,\omega^{\,\mu\nu}$:

$$\delta x^{\mu} = \delta\,\omega^{\,\mu} + x_{\,\nu}\,\delta\,\omega^{\,\mu\nu} \qquad \delta\,\omega^{\,\mu\nu} = -\,\delta\,\omega^{\,\nu\mu}$$

In general we shall suppose to deal with a collection of N field functions

$$u_A(x) \qquad (A = 1, 2, \ldots, N)$$

with a well defined infinitesimal transformation law under the continuous symmetry (2.86) that may be written in the form

$$u_A(x) \quad \mapsto \quad u_A^{\prime}(x^{\prime}) = u_A(x) + \Delta\,u_A(x)$$

$$\Delta\,u_A(x) = u_A^{\prime}(x^{\,\prime}\,) - u_A(x) = Y_{AB}^{\,a}\,u_B\,(x)\;\delta\,\omega^{\,a} \tag{2.87}$$

where the very last variation is just the already introduced total variation (2.2) of the field function. For example, in the case of an infinitesimal Lorentz transformation, the definitions (2.25), (2.42), (2.43), (2.72) and (2.73) yield

$$\Delta\, u_A(x) = Y^{\,a}_{AB}\, u_B\,(x)\ \delta\,\omega^{\,a} \ = \ -\,\tfrac{1}{2}\,\mathrm{i}\,(S_{\mu\nu})_{AB}\, u_B\,(x)\ \delta\,\omega^{\,\mu\nu} \tag{2.88}$$

The jacobian that does correspond to the infinitesimal transformation of coordinates is

$$\delta J \ = \ \delta \det \parallel \partial x'^{\,\mu}/\partial x^{\,\nu} \parallel = \ \partial_\mu\, \delta x^{\,\mu}$$

so that we can eventually write

$$\begin{aligned}
\Delta S \ &= \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)\, \partial_\lambda\, \delta x^{\,\lambda} \\
&+ \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \Delta\, \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big) \\
&= \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)\, \partial_\lambda\, \delta x^{\,\lambda} \\
&+ \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \partial_\lambda\, \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)\, \delta x^{\,\lambda} \\
&+ \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \delta\mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big)
\end{aligned}$$

where use has been made of the relation (2.15). If the Euler-Lagrange wave field equations (2.85) are assumed to be valid besides the radial asymptotic behaviour (2.84), then we can recast the local variation of the lagrangian in the form

$$\delta\, \mathcal{L}\Big(u_A(x), \partial_\mu u_A(x)\Big) = \partial_\mu \left[\, \frac{\delta \mathcal{L}}{\delta\, \partial_\mu\, u_A(x)}\ \delta u_A(x) \right]$$

Hence we immediately obtain

$$\Delta S \ \ = \ \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\ \partial_\mu \left[\, \mathcal{L}(x)\, \delta x^{\,\mu} + \frac{\delta \mathcal{L}}{\delta\, \partial_\mu\, u_A(x)}\ \delta u_A(x) \right] \tag{2.89}$$

Alternatively, we can always recast the local variations (2.3) in terms of the total variations (2.2) so that

$$\begin{aligned}
\Delta S \ \ &= \ \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \partial_\mu \left\{ \left[\, \delta^{\,\mu}_{\,\nu}\, \mathcal{L}(x) \ - \ \frac{\delta \mathcal{L}}{\delta\, \partial_\mu u_A(x)}\ \partial_\nu\, u_A(x) \right]\, \delta x^{\,\nu} \right\} \\
&+ \ \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\mathbf{x}\ \partial_\mu \left[\, \frac{\delta \mathcal{L}}{\delta\, \partial_\mu u_A(x)}\ \Delta u_A(x) \right]
\end{aligned} \tag{2.90}$$

Consider now the infinitesimal symmetry transformations depending upon constant parameters, *i.e.* spacetime independent, so that $\delta\,\omega^{\,a} = \Delta\,\omega^{\,a}$:

$$\delta x^{\,\mu} \equiv \left(\frac{\partial\, x^{\,\mu}}{\partial\,\omega^{\,a}}\right)\,\delta\,\omega^{\,a} = X_a^{\,\mu}\,\delta\,\omega^{\,a} \qquad (a = 1,2,\ldots,s)$$

$$\Delta\,u_A(x) \equiv (\,Y_a\,)_{\,AB}\; u_B\,(x)\;\delta\,\omega^{\,a} \qquad (A = 1,2,\ldots,N)$$

then we have

$$\frac{\Delta S}{\Delta\,\omega^{\,a}} \;+\; \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\;\partial_\mu J_a^{\,\mu}(x) \;=\; 0 \tag{2.91}$$

where

$$\begin{aligned} J_a^{\,\mu}(x) \;&\equiv\; \left[\,\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\;\partial_\nu\,u_A(x) \;-\; \mathcal{L}(x)\,\delta_\nu^{\,\mu}\,\right] X_a^{\,\nu} \\ &-\; \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\;(\,Y_a\,)_{\,AB}\; u_B\,(x) \end{aligned} \tag{2.92}$$

are the Noether currents associated to the parameters $\omega^{\,a}$ $(a = 1,2,\ldots,s)$ of the Lie group of global symmetry transformations. Suppose the action functional to be invariant under this group of global transformations

$$(\Delta S\,/\,\Delta\,\omega^{\,a}) = 0 \qquad (a = 1,2,\ldots,s)$$

Then from Gauß theorem we get

$$\begin{aligned} 0 \;&=\; \int \mathrm{d}\mathbf{x}\,\Big[\,J_a^{\,0}(t_f,\mathbf{x}) - J_a^{\,0}(t_i,\mathbf{x})\Big] + \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\mathbf{x}\;\boldsymbol{\nabla}\cdot\mathbf{J}\,(x_0,\mathbf{x}) \\ &=\; \int \mathrm{d}\mathbf{x}\,\Big[\,J_a^{\,0}(t_f,\mathbf{x}) - J_a^{\,0}(t_i,\mathbf{x})\Big] \\ &+\; \lim_{R\to\infty}\; R^{\,2} \int_{t_i}^{t_f} \mathrm{d}x_0 \int \mathrm{d}\Omega\;\widehat{\mathbf{r}}\cdot\mathbf{J}_a\,(x_0,R,\Omega) \end{aligned}$$

where R is the ray of a very large sphere centered at $\mathbf{x} = 0$, $\Omega = (\theta,\phi)$ is the solid angle in the three-dimensional space and $\widehat{\mathbf{r}}$ is the exterior unit normal vector to the sphere, *i.e.* the radial unit vector. Once again, if we assume the radial asymptotic behaviour

$$\lim_{R\to\infty}\; R^{\,2}\;\widehat{\mathbf{r}}\cdot\mathbf{J}_a(x_0,R,\Omega) = 0 \qquad \forall x_0 \in [\,t_i\,,\,t_f\,] \tag{2.93}$$

then the above boundary term indeed disappears and we eventually come to the conservation laws

$$\frac{\Delta S}{\Delta\,\omega^{\,a}} = 0 \qquad \Leftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,Q_a(t) \;=\; 0 \tag{2.94}$$

where the conserved Noether charges are defined to be

$$Q_a \;\equiv\; \int \mathrm{d}\mathbf{x}\; J_a^0\,(x) \qquad (a = 1, 2, \ldots, s) \tag{2.95}$$

This is Noether theorem. Notice that the Noether current is not univocally identified : as a matter of fact, if we redefine the Noether currents (2.92) according to

$$\tilde{J}_a^{\,\mu}\,(x) \equiv J_a^{\,\mu}\,(x) + \partial_\nu\, \mathcal{A}_a^{\,\mu\nu}\,(x)$$

where $\mathcal{A}_a^{\mu\nu}\,(x)$ $(a = 1, 2, \ldots, s)$ is an arbitrary set of antisymmetric tensor fields

$$\mathcal{A}_a^{\,\mu\nu}\,(x) + \mathcal{A}_a^{\,\nu\mu}\,(x) = 0$$

then by construction

$$\partial_\mu\, \tilde{J}_a^{\,\mu}\,(x) = \partial_\mu\, J_a^{\,\mu}\,(x) = \partial_t\, J_a^{\,0}(t, \mathbf{r}) + \boldsymbol{\nabla} \cdot \mathbf{J}_a(t, \mathbf{r})$$

and the same conserved Noether charges (2.95) are obtained. Let us now examine some important examples.

1. Spacetime translations

$$\delta x^{\,\mu} = \delta\, \omega^{\,\mu} \qquad a \equiv \rho = 0, 1, 2, 3$$

$$X_a^{\,\nu} \equiv \delta_\rho^{\,\nu} \qquad \Delta\, u_A(x) \equiv 0$$

because there is no change under spacetime translations for any classical relativistic wave field. In this case the corresponding Noether's current yields the *energy momentum tensor*

$$J_a^{\,\mu}\,(x) \quad \mapsto \quad T^{\,\mu}{}_{\rho}\,(x)$$

$$T^{\,\mu\rho}\,(x) \;=\; \frac{\delta \mathcal{L}}{\delta\, \partial_\mu\, u_A(x)}\; \partial^\rho u_A(x) \;-\; \mathcal{L}(x)\, g^{\,\mu\rho} \tag{2.96}$$

the corresponding conserved charge being the total energy momentum four vector of the system

$$P_\mu = \int \mathrm{d}\mathbf{x}\; T_\mu^{\,0}\,(x) \tag{2.97}$$

2. Lorentz transformations

$$\delta x^{\nu} = \delta\omega^{\nu\rho} x_{\rho} \qquad a \equiv \{\rho\sigma\} = 1,\ldots,6$$

and from (2.88)

$$X_a^{\nu} \equiv \tfrac{1}{2}\left(x_{\sigma}\,\delta_{\rho}^{\nu} - x_{\rho}\,\delta_{\sigma}^{\nu}\right) \qquad \Delta\, u_A(x) \equiv -\tfrac{1}{2}\,\mathrm{i}\,(S_{\rho\sigma})_{AB}\, u_B(x)\,\delta\omega^{\rho\sigma}$$

In this case the corresponding Noether current yields the relativistic *total angular momentum density* third rank tensor

$$J_a^{\mu}(x) = J^{\mu}{}_{\rho\sigma}(x) = -\tfrac{1}{2}\,M^{\mu}{}_{\rho\sigma}(x)$$

$$\begin{aligned} M^{\mu}{}_{\rho\sigma}(x) &= x_{\rho}\,T^{\mu}{}_{\sigma}(x) - x_{\sigma}\,T^{\mu}{}_{\rho}(x) \\ &- \mathrm{i}\,\frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\,u_A(x)}\,(S_{\rho\sigma})_{AB}\,u_B(x) \\ &\stackrel{\mathrm{def}}{=} L^{\mu}{}_{\rho\sigma}(x) + S^{\mu}{}_{\rho\sigma}(x) \end{aligned} \tag{2.98}$$

where the third rank tensors

$$L^{\mu}{}_{\rho\sigma}(x) = -\,L^{\mu}{}_{\sigma\rho}(x) = x_{\rho}\,T^{\mu}{}_{\sigma}(x) - x_{\sigma}\,T^{\mu}{}_{\rho}(x)$$

$$S^{\mu}{}_{\rho\sigma}(x) = -\,S^{\mu}{}_{\sigma\rho}(x) = (-\,\mathrm{i}\,)\,\frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\,u_A(x)}\,(S_{\rho\sigma})_{AB}\,u_B(x)$$

are respectively the relativistic orbital angular momentum tensor and the relativistic spin angular momentum tensor of the wave field. The corresponding charge is the total angular momentum antisymmetric tensor of the system

$$M_{\mu\nu} = \int \mathrm{d}\mathbf{x}\; M^{0}{}_{\mu\nu}(t,\mathbf{x}) \qquad M_{\mu\nu} + M_{\nu\mu} = 0 \tag{2.99}$$

Notice however that, as we shall see further on, in the case of Lorentz boost transformations the angular momentum density tensor does not satisfy the condition (2.93) owing to its explicit time dependence so that, consequently, the related charge is not conserved in time.

3. Internal symmetries

$$X_a^{\nu} \equiv 0 \qquad \Delta\, u_A(x) \equiv -\,T_{AB}^{a}\,u_B(x) \qquad (a = 1,2,\ldots,s) \tag{2.100}$$

where T^{a} $(a = 1,2,\ldots,s)$ are the generators of the internal symmetry Lie group in some given representation. In this case the corresponding

Noether current and charge yields the *internal symmetry current and charge multiplets*

$$J_\mu^a(x) = \frac{\delta \mathcal{L}}{\delta\, \partial_\mu\, u_A(x)}\, T_{AB}^a\, u_B\,(x) \qquad (a = 1, 2, \ldots, s) \tag{2.101}$$

$$Q^a = \int \mathrm{d}\mathbf{x}\; J_0^a(x) \qquad (a = 1, 2, \ldots, s) \tag{2.102}$$

References

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2.3.1 Problems

1. Construct the energy momentum the total angular momentum tensor densities for classical electromagnetism with no sources, *i.e.* for the classical radiation field. [2]

   Solution. The classical Maxwell's lagrangian is given by

   $$\mathcal{L}(x) = -\,\tfrac{1}{4}\, F^{\,\mu\nu}\,(x)\, F_{\mu\nu}\,(x) \qquad F_{\mu\nu}\,(x) = \partial_\mu\, A_\nu\,(x) - \partial_\nu\, A_\mu\,(x)$$

   hence from Noether theorem we get the canonical energy momentum tensor

   $$\begin{aligned} T^{\,\mu\rho}\,(x) \;&=\; \frac{\delta \mathcal{L}}{\delta\, \partial_\mu\, A_\nu(x)}\; \partial^{\,\rho} A_\nu(x) \;-\; \mathcal{L}(x)\, g^{\mu\rho} \\ &=\; -\, F^{\,\mu\nu}(x)\; \partial^{\,\rho} A_\nu(x) + \tfrac{1}{4}\, F^{\,\lambda\nu}(x)\, F_{\lambda\nu}(x)\, g^{\mu\rho} \end{aligned}$$

[2] See *e.g.* L.D. Landau, E.M. Lifšits, *Teoria dei campi*, Editori Riuniti/Edizioni Mir, Roma, 1976, § 33 p. 114.

which is not symmetric with respect to μ and ρ. To remedy this, in accordance with the general rule, we shall introduce

$$\Theta^{\mu\rho}(x) = T^{\mu\rho}(x) + \partial_\nu \Big(A^\rho(x) F^{\mu\nu}(x) \Big)$$

and thereby, using the field equations $\partial_\nu F^{\mu\nu}(x) = 0$, we can write

$$\begin{aligned} \Theta^{\mu\rho}(x) &= T^{\mu\rho}(x) + F^{\mu\nu}(x)\, \partial_\nu A^\rho(x) \\ &= \tfrac{1}{4} F^{\lambda\nu}(x) F_{\lambda\nu}(x)\, g^{\mu\rho} - F^{\mu\nu}(x) F^{\rho}{}_{\nu}(x) = \Theta^{\rho\mu}(x) \end{aligned}$$

Notice that the above obtained symmetric energy momentum tensor of the electromagnetic field is traceless

$$g_{\mu\rho}\Theta^{\mu\rho}(x) = 0$$

Let us express explicitly the well known components :

$$\begin{aligned} &\Theta_{jk} = \tfrac{1}{2}\delta_{jk}(\mathbf{E}^2 + \mathbf{B}^2) - E_j E_k - B_j B_k \quad \text{(Maxwell stress tensor)} \\ &\Theta^{0j} = S^j = \varepsilon^{jkl} E^k B^l \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B} \qquad \text{(Poynting vector)} \\ &\Theta_{00} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \qquad \text{(energy density)} \end{aligned}$$

From equations (2.23) and (2.98) we derive the canonical total angular momentum density of the radiation field

$$\begin{aligned} M^{\mu}{}_{\rho\sigma} &= x_\rho T^{\mu}{}_{\sigma} - x_\sigma T^{\mu}{}_{\rho} + i F^{\mu\nu}(S_{\rho\sigma})_{\nu\lambda} A^\lambda \\ &= x_\rho \Big(\Theta^{\mu}{}_{\sigma} - F^{\mu\nu}\partial_\nu A_\sigma \Big) - x_\sigma \Big(\Theta^{\mu}{}_{\rho} - F^{\mu\nu}\partial_\nu A_\rho \Big) \\ &- F^{\mu}{}_{\rho} A_\sigma + F^{\mu}{}_{\sigma} A_\rho \\ &= x_\rho \Theta^{\mu}{}_{\sigma} - x_\sigma \Theta^{\mu}{}_{\rho} - \partial_\nu \left[F^{\mu\nu}(x_\rho A_\sigma - x_\sigma A_\rho) \right] \end{aligned}$$

The very last term does not contribute at all to the continuity equation by virtue of the antisymmetry with respect to the pair of indices $\mu\nu$. On the one hand, it turns out to be manifest that the total angular momentum tensor can always appear to be of a purely orbital form. On the other hand, it is clear that the spin angular momentum second rank tensor of the radiation field is not conserved in time.

Chapter 3

The Scalar Field

3.1 Normal Modes Expansion

The simplest though highly nontrivial example of a quantum field theory involves a real scalar field $\phi : \mathcal{M} \to \mathbb{R}$. The most general Poincaré invariant lagrangian density that fulfill all the criteria listed in section 2.2 takes the general form

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi(x)\,\partial^\mu \phi(x) - V\,[\,\phi(x)\,] \tag{3.1}$$

where $V(\phi)$ is assumed to be a real analytic function of its argument, that is

$$V\,[\,\phi(x)\,] = \frac{1}{2}\,m^2\,\phi^2(x) + \frac{g}{3!}\,m\,\phi^3(x) + \frac{\lambda}{4!}\,\phi^4(x) + \cdots \tag{3.2}$$

with $m > 0$ and $g, \lambda \in \mathbb{R}$. Notice that the structure of the kinetic term is such to fix the canonical dimension of the scalar field : $[\,\phi\,] = \mathrm{cm}^{-1} = \mathrm{eV}$ in natural units $\hbar = c = 1$.

The Euler-Lagrange equations of motion read

$$\Box\phi(x) + m^2\,\phi(x) + \frac{1}{2}\,mg\,\phi^2(x) + \frac{\lambda}{6}\,\phi^3(x) + \cdots = 0 \tag{3.3}$$

while the conserved energy momentum tensor (2.96) and vector (2.97) are

$$T_{\mu\nu}\,(x) = \partial_\mu \phi(x)\,\partial_\nu \phi(x) - g_{\mu\nu}\,\mathcal{L}(x) = T_{\nu\mu}\,(x) \tag{3.4}$$

$$P_\mu = \int \mathrm{d}\mathbf{x}\;T^{\,0}_{\mu}\,(x) = \int \mathrm{d}\mathbf{x}\,\left[\partial_0\,\phi(x)\partial_\mu\phi(x) - g_{0\mu}\,\mathcal{L}(x)\right] \tag{3.5}$$

It is very easy to check that thanks to the Euler–Lagrange equations of motion the energy momentum current is conserved, *i.e.*

$$\partial_\mu T^{\mu}{}_{\nu}(x) = 0$$

Moreover, the stability requirement that the total energy must be bounded from below, to avoid a colapse of the mechanical system, entails that the analytic potential has to be an *even and concave functional* of the real scalar field so that $g = 0\,,\ \lambda > 0$ *et cetera*. Finally, as it will be better focussed later on, the constraining *criterion of power counting renormalizability* for the corresponding quantum field theory will forbid the presence of coupling parameters with canonical dimensions equal to positive integer powers of length. In such a circumstance, the Lagrange density for a self–interacting, stable, renormalizable real scalar field theory reduces to

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu}\,\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}\, m^2\,\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x) \tag{3.6}$$

whence we obtain the canonical momentum

$$\Pi(x) \;\equiv\; \frac{\delta L}{\delta\partial_0\phi(x)} = \frac{\delta S}{\delta\partial_0\phi(x)} = \partial_0\,\phi(x) = \dot{\phi}(x) \tag{3.7}$$

$$L(x_0) \;\equiv\; \int \mathrm{d}\mathbf{x}\,\mathcal{L}(x) \qquad S \equiv \int_{t_i}^{t_f} \mathrm{d}x_0\; L(x_0) \tag{3.8}$$

the hamiltonian

$$\begin{aligned} H &\equiv P^0 = \int \mathrm{d}\mathbf{x}\; T^{0}_{\;0}(x^0,\mathbf{x}) = H_0 + H_I \geq 0 \\ H_0 &= \frac{1}{2}\int \mathrm{d}\mathbf{x}\left[\Pi^2(x) - \phi(x)\boldsymbol{\nabla}^2\phi(x) + m^2\,\phi^2(x)\right] \end{aligned} \tag{3.9}$$

$$H_I = \int \mathrm{d}\mathbf{x}\;\frac{\lambda}{4!}\,\phi^4(x) \tag{3.10}$$

the total momentum

$$\mathbf{P} \equiv -\int \mathrm{d}\mathbf{x}\;\Pi(x)\,\boldsymbol{\nabla}\phi(x) \tag{3.11}$$

and the total orbital angular momentum

$$L_{\mu\nu} = \int \mathrm{d}\mathbf{x}\;\left[x_\mu\,T_{0\nu}(t,\mathbf{x}) - x_\nu\,T_{0\mu}(t,\mathbf{x})\right] \tag{3.12}$$

We observe *en passant* that the above expressions are indeed obtained by making use of the asymptotic radial behaviour

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|\,\phi(x_0,\mathbf{x})\,\mathbf{x}\cdot\boldsymbol{\nabla}\phi(x_0,\mathbf{x}) = 0$$

by virtue of which and of the Gauß theorem we can write

$$\int \mathrm{d}\mathbf{x}\ \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) = -\int \mathrm{d}\mathbf{x}\ \phi(x)\,\boldsymbol{\nabla}^2\phi(x)$$

It is also worthwhile to remark that the orbital angular momentum tensor

$$M^{\lambda}{}_{\mu\nu}(x) = x_{\mu}\,T^{\lambda}{}_{\nu}(x) - x_{\nu}\,T^{\lambda}{}_{\mu}(x)$$

does fulfill the continuity equation thanks to the symmetry of the energy momentum tensor $T_{\rho\sigma}(x)$.

The equations of motion can also be written in the hamiltonian canonical form. From the fundamental canonical Poisson's brackets

$$\{\phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \delta\,(\mathbf{x}-\mathbf{y})$$

$$\{\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})\} = 0 = \{\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\}$$

one can immediately get

$$\begin{cases} \dot{\phi}(x) = \{\,\phi\,(t,\mathbf{x})\,,\,H\,\} = \delta\,H/\delta\,\Pi\,(x) \\ \dot{\Pi}\,(x) = \{\,\Pi\,(t,\mathbf{x})\,,\,H\,\} = -\,\delta\,H/\delta\,\phi\,(x) \end{cases}$$

and thereby

$$\dot{\phi}(x) = \Pi\,(x) \tag{3.13}$$

$$\dot{\Pi}\,(x) = \ddot{\phi}(x) = \boldsymbol{\nabla}^2\phi(x) - m^2\,\phi(x) - \frac{\lambda}{3!}\,\phi^3(x) \tag{3.14}$$

Consider the *free scalar field theory* in which the action, the lagrangian and the hamiltonian are quadratic functional of the scalar field function, so that the Euler–Lagrange equations of motion becomes linear and exactly solvable: namely,

$$\mathcal{L}_0 = \tfrac{1}{2}\,g^{\mu\nu}\,\partial_{\mu}\phi\,(x)\,\partial_{\nu}\phi\,(x) - \tfrac{1}{2}\,m^2\,\phi^2(x) \tag{3.15}$$

$$H_0 = \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\left[\,\Pi^2(x) - \phi\,(x)\boldsymbol{\nabla}^2\phi\,(x) + m^2\,\phi^2(x)\,\right] \tag{3.16}$$

$$\left(\Box + m^2\right)\,\phi(x) = 0 \tag{3.17}$$

which is the *Klein-Gordon* relativistic wave equation. To solve it, let us introduce the Fourier decomposition

$$\phi(x) \equiv (2\pi)^{-3/2} \int \mathrm{d}k\ \tilde{\phi}(k)\ \exp\{-\,\mathrm{i}\,k\cdot x\} \qquad \tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{3.18}$$

so that

$$(k^2 - m^2)\,\tilde{\phi}(k) = 0$$

the most general solution of which reads

$$\tilde{\phi}(k) = f(k)\,\delta(k^2 - m^2)$$

$f(k)$ being an arbitrary complex function which is regular on the momentum space hyperbolic manifold $k^2 = m^2$ ($k \in \mathcal{M}$) and which fulfill the *reality condition* $f(k) = f(-k)$ in order the scalar field to be real. Then we have

$$\phi(x) \equiv (2\pi)^{-3/2} \int \mathrm{d}k\ f(k)\,\delta(k^2 - m^2)\ \exp\{-\,\mathrm{i}\,k\cdot x\} \tag{3.19}$$

and from the well known tempered distribution identities

$$\delta(ax) = \frac{1}{|a|}\,\delta(x) \qquad \theta(x) + \theta(-x) = 1 \tag{3.20}$$

one can readily obtain the decomposition

$$\delta(k^2 - m^2) = \frac{1}{2\omega_{\mathbf{k}}}\,[\,\theta(k_0)\,\delta(k_0 - \omega_{\mathbf{k}}) + \theta(-\,k_0)\,\delta(k_0 + \omega_{\mathbf{k}})\,]$$

$$\omega_{\mathbf{k}} = \omega(\mathbf{k}) \equiv (\mathbf{k}^2 + m^2)^{\frac{1}{2}} = \omega_{-\mathbf{k}} \tag{3.21}$$

Hence

$$\begin{aligned}
\phi(x) &= (2\pi)^{-3/2} \int \mathrm{d}k\ f(k)\ \exp\{-\,\mathrm{i}\,k\cdot x\} \\
&\times\ \frac{1}{2\omega_{\mathbf{k}}}\,[\,\theta(k_0)\,\delta(k_0 - \omega_{\mathbf{k}}) + \theta(-\,k_0)\,\delta(k_0 + \omega_{\mathbf{k}})\,] \\
&= (2\pi)^{-3/2} \int \frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}}\ f(\omega_{\mathbf{k}}, \mathbf{k})\ \exp\{-\,\mathrm{i}\,x^0\omega_{\mathbf{k}} + \mathrm{i}\mathbf{k}\cdot\mathbf{x}\} \\
&+\ (2\pi)^{-3/2} \int \frac{\mathrm{d}\mathbf{k}}{2\omega_{\mathbf{k}}}\ f(-\omega_{\mathbf{k}}, -\mathbf{k})\ \exp\{\mathrm{i}\,x^0\omega_{\mathbf{k}} - \mathrm{i}\mathbf{k}\cdot\mathbf{x}\} \\
&= \int \frac{\mathrm{d}\mathbf{k}}{[\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{1/2}} \cdot \frac{f(\omega_{\mathbf{k}}, \mathbf{k})}{\sqrt{(2\omega_{\mathbf{k}})}}\ \exp\{-\,\mathrm{i}\,x^0\omega_{\mathbf{k}} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\} + \text{c.c.}
\end{aligned}$$

where we used the reality condition. It is very convenient to set

$$u_{\mathbf{k}}(x) \equiv [\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{-1/2}\ \exp\{-\,\mathrm{i}\,x^0\omega_{\mathbf{k}} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\} \tag{3.22}$$

so that

$$\begin{aligned}\phi(x) &= \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}\, u_{\mathbf{k}}(x) + f_{\mathbf{k}}^*\, u_{\mathbf{k}}^*(x) \Big] \\ \Pi(x) &= \sum_{\mathbf{k}} \mathrm{i}\,\omega_{\mathbf{k}} \Big[\, f_{\mathbf{k}}^*\, u_{\mathbf{k}}^*(x) - f_{\mathbf{k}}\, u_{\mathbf{k}}(x) \Big]\end{aligned} \tag{3.23}$$

with the suitable notations

$$f_{\mathbf{k}} \equiv \frac{f(\omega_{\mathbf{k}},\mathbf{k})}{\surd(2\omega_{\mathbf{k}})} \qquad\qquad \int \mathrm{d}\mathbf{k} \ \equiv \ \sum_{\mathbf{k}}$$

which endorse the fact that (3.23) is nothing but the *normal mode expansion* of the real scalar field. The wave functions $u_{\mathbf{k}}(x)$ ($\mathbf{k} \in \mathbb{R}$) do constitute a complete and orthonormal set, *i.e.* the normal modes of the real scalar free field : namely, they fulfill the orthonormality relations

$$\int \mathrm{d}\mathbf{x}\ u_{\mathbf{h}}^*(x)\,\mathrm{i}\,\overleftrightarrow{\partial}_0\, u_{\mathbf{k}}(x) = \delta(\mathbf{h}-\mathbf{k}) \tag{3.24}$$

$$\int \mathrm{d}\mathbf{x}\ u_{\mathbf{h}}(x)\,\mathrm{i}\,\overleftrightarrow{\partial}_0\, u_{\mathbf{k}}(x) = \int \mathrm{d}\mathbf{x}\ u_{\mathbf{h}}^*(x)\,\mathrm{i}\,\overleftrightarrow{\partial}_0\, u_{\mathbf{k}}^*(x) = 0 \tag{3.25}$$

as well as the closure relation

$$\sum_{\mathbf{k}}\ u_{\mathbf{k}}(x)\, u_{\mathbf{k}}^*(y) = \frac{1}{\mathrm{i}}\, D^{\,(-)}\,(x-y) \tag{3.26}$$

where I have set

$$D^{\,(-)}\,(x-y) \stackrel{\mathrm{def}}{=} \mathrm{i} \int \frac{dk}{(2\pi)^3}\ \theta\,(k_0\,)\ \delta\left(k^{\,2}-m^{\,2}\right)\ \exp\{-\,\mathrm{i}\,k\cdot(x-y)\}$$

From these orthonormality relations it is easy to invert the normal mode expansions that yields

$$\int \mathrm{d}\mathbf{x}\ u_{\mathbf{k}}^*(x)\,\mathrm{i}\,\overleftrightarrow{\partial}_0\, \phi(x) = f_{\mathbf{k}} \tag{3.27}$$

As it is well known the normal mode decomposition is what we need to diagonalize the energy momentum vector of the mechanical system. As a matter of fact, from the equalities

$$\begin{aligned}
P_0 &= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2}\left[\Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x)\right] \\
&= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2}\left\{\dot\phi(x)\,\dot\phi(x) - \phi(x)\left[\boldsymbol{\nabla}^2\phi(x) - m^2\,\phi(x)\right]\right\} \\
&= \int \mathrm{d}\mathbf{x}\ \tfrac{1}{2}\left\{\dot\phi^2(x) - \phi(x)\,\ddot\phi(x)\right\} \qquad (3.28)\\
\mathbf{P} &= -\int \mathrm{d}\mathbf{x}\ \Pi(x)\boldsymbol{\nabla}\phi(x) \qquad (3.29)
\end{aligned}$$

by substituting the normal mode expansions (3.23) we immediately obtain, for instance,

$$\begin{aligned}
\mathbf{P} &= -\int \mathrm{d}\mathbf{x}\ \sum_{\mathbf{k}} i\,\omega_{\mathbf{k}}\Big[\,f^*_{\mathbf{k}}\,u^*_{\mathbf{k}}(x) - f_{\mathbf{k}}\,u_{\mathbf{k}}(x)\Big] \\
&\times\ \sum_{\mathbf{h}}\Big[\,i\,\mathbf{h}\,f_{\mathbf{h}}\,u_{\mathbf{h}}(x) - i\,\mathbf{h}\,f^*_{\mathbf{h}}\,u^*_{\mathbf{h}}(x)\Big]
\end{aligned}$$

and from the orthonormality relations we readily get

$$\begin{aligned}
\mathbf{P} &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\mathbf{k}\Big(f_{\mathbf{k}}f^*_{\mathbf{k}} + f^*_{\mathbf{k}}f_{\mathbf{k}}\Big) \qquad (3.30)\\
P_0 &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\omega_{\mathbf{k}}\Big(f_{\mathbf{k}}f^*_{\mathbf{k}} + f^*_{\mathbf{k}}f_{\mathbf{k}}\Big) \qquad (3.31)
\end{aligned}$$

The normal mode complex amplitudes actually share the fractional canonical dimensions $[\,f_{\mathbf{k}}\,] = \mathrm{cm}^{3/2} = \mathrm{eV}^{-3/2}$. The complex amplitudes of the normal modes are also called the *holomorphic coordinates* of the real scalar free field. It is also quite evident that by introducing the related real *canonical coordinates*

$$\begin{cases} Q_{\mathbf{k}} \equiv \quad (2\omega_{\mathbf{k}})^{-1/2}\,(f_{\mathbf{k}} + f^*_{\mathbf{k}}) \\ P_{\mathbf{k}} \equiv \ -\mathrm{i}\,\left(\tfrac{1}{2}\,\omega_{\mathbf{k}}\right)^{1/2}(f_{\mathbf{k}} - f^*_{\mathbf{k}}) \end{cases}$$

$$f_{\mathbf{k}} = \sqrt{\tfrac{1}{2}}\left(\omega_{\mathbf{k}}^{\frac{1}{2}}\,Q_{\mathbf{k}} + \mathrm{i}\,\omega_{\mathbf{k}}^{-\frac{1}{2}}\,P_{\mathbf{k}}\right) \qquad (3.32)$$

we can write eventually

$$\begin{aligned}
P_0 = \sum_{\mathbf{k}} H_{\mathbf{k}} &= \sum_{\mathbf{k}} \tfrac{1}{2}\left(P^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}Q^2_{\mathbf{k}}\right) \\
&= \sum_{\mathbf{k}} \tfrac{1}{2}\,\omega_{\mathbf{k}}\Big(f_{\mathbf{k}}f^*_{\mathbf{k}} + f^*_{\mathbf{k}}f_{\mathbf{k}}\Big) \qquad (3.33)
\end{aligned}$$

with $[\,P_{\mathbf{k}}\,] = \mathrm{cm}\,,\ [\,Q_{\mathbf{k}}\,] = \mathrm{cm}^2$ which explicitly shows that a real scalar field is dynamically fully equivalent to an assembly of an infinite number of decoupled harmonic oscillators of principal frequencies $\omega_{\mathbf{k}} = (\mathbf{k}^2 + m^2)^{1/2}$.

As a matter of fact, we can rewrite the field and conjugated momentum expansions (3.23) in the suggestive form

$$\begin{aligned}\phi(x) &= \sum_{\mathbf{k}} \Big[\, f_{\mathbf{k}}(t)\, u_{\mathbf{k}}(\mathbf{x}) + f^*_{\mathbf{k}}(t)\, u^*_{\mathbf{k}}(\mathbf{x}) \Big] \\ \Pi(x) &= \sum_{\mathbf{k}} \mathrm{i}\,\omega_{\mathbf{k}} \Big[\, f^*_{\mathbf{k}}(t)\, u^*_{\mathbf{k}}(\mathbf{x}) - f_{\mathbf{k}}(t)\, u_{\mathbf{k}}(\mathbf{x}) \Big]\end{aligned} \tag{3.34}$$

where of course

$$\begin{aligned}& f_{\mathbf{k}}(t) = f_{\mathbf{k}}\, \exp\{-\mathrm{i}\omega_{\mathbf{k}} t\} \qquad f^*_{\mathbf{k}}(t) = f_{\mathbf{k}}\, \exp\{\mathrm{i}\omega_{\mathbf{k}} t\} \\ & u_{\mathbf{k}}(\mathbf{x}) = [\, 2\omega_{\mathbf{k}} (2\pi)^3\,]^{-1/2}\ \exp\{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\}\end{aligned} \tag{3.35}$$

and from the standard canonical Poisson's brackets [1] for the linear oscillator

$$\{Q_{\mathbf{h}}\,,\, Q_{\mathbf{k}}\} = 0 \qquad \{P_{\mathbf{h}}\,,\, P_{\mathbf{k}}\} = 0$$

$$\{Q_{\mathbf{h}}\,,\, P_{\mathbf{k}}\} = \delta\,(\,\mathbf{h} - \,\mathbf{k})$$

from (3.32) it is very simple to derive the canonical hamiltonian equations for the time dependent holomorphic coordinates $f_{\mathbf{k}}(t)$: namely,

$$\{f_{\mathbf{h}}\,,\, f_{\mathbf{k}}\} = 0 = \{f^*_{\mathbf{h}}\,,\, f^*_{\mathbf{k}}\} \qquad \{f_{\mathbf{h}}\,,\, f^*_{\mathbf{k}}\} = -\,\mathrm{i}\,\delta(\mathbf{h} - \mathbf{k}) \tag{3.36}$$

$$\dot{f}_{\mathbf{k}}(t) = \{f_{\mathbf{k}}(t)\,,\, H\} = -\,\mathrm{i}\omega_{\mathbf{k}}\, f_{\mathbf{k}}(t) \tag{3.37}$$

the solution of which is just provided by (3.35).

[1] We recall the definition of the Poisson's brackets for two differentiable functions F and G on the phase space of a mechanical system

$$\{F\,,\, G\} \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\,\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\,\frac{\partial F}{\partial p_i} \right)$$

3.2 Quantization of a Klein-Gordon Field

Once that the dynamical treatment of the free real scalar field has been developed within the canonical hamiltonian formulation, the quantization of the system will directly follow in accordance with the Dirac *correspondence principle* – see any textbook of quantum mechanics *e.g.* [4]. According to the quantization rules for a linear hamonic oscillator, we shall introduce for each normal mode of the real scalar field the corresponding linear operators acting on the related Hilbert space and the associated algebra, *i.e.*

$$Q_{\mathbf{k}} \longmapsto \widehat{Q}_{\mathbf{k}} = \widehat{Q}_{\mathbf{k}}^{\dagger} \qquad P_{\mathbf{k}} \longmapsto \widehat{P}_{\mathbf{k}} = \widehat{P}_{\mathbf{k}}^{\dagger}$$

$$\widehat{P}_{\mathbf{k}} = \frac{-\,\mathrm{i}\,\hbar\,\mathrm{d}}{\mathrm{d}Q_{\mathbf{k}}}$$

$$\{Q_{\mathbf{h}}\,,\,P_{\mathbf{k}}\} \longmapsto \frac{1}{\mathrm{i}\hbar}\,[\,\widehat{Q}_{\mathbf{h}}\,,\,\widehat{P}_{\mathbf{k}}\,]$$

$$H_{\mathbf{k}} \longmapsto \widehat{H}_{\mathbf{k}} = \tfrac{1}{2}\,\widehat{P}_{\mathbf{k}}^{\,2} + \tfrac{1}{2}\,\omega_{\mathbf{k}}^{2}\,\widehat{Q}_{\mathbf{k}}^{\,2} \tag{3.38}$$

and the Poisson's brackets among the holomorphic coordinates turn into the commutators among the creation distruction operators

$$f_{\mathbf{k}} \longmapsto a_{\mathbf{k}} \qquad f_{\mathbf{k}}^{*} \longmapsto a_{\mathbf{k}}^{\dagger} \tag{3.39}$$

$$[\,a_{\mathbf{h}}\,,\,a_{\mathbf{k}}\,] = 0 \qquad [\,a_{\mathbf{h}}^{\dagger}\,,\,a_{\mathbf{k}}^{\dagger}\,] = 0 \tag{3.40}$$

$$[\,a_{\mathbf{h}}\,,\,a_{\mathbf{k}}^{\dagger}\,] = \delta(\mathbf{h}-\mathbf{k}) \tag{3.41}$$

As a consequence the scalar field function $\phi(x)$ together with its conjugated momentum $\Pi(x)$ will turn, after quantization, into *operator valued tempered distributions*, the normal mode expansions of which can be obtained in a straightforward way from (3.23) and (3.39), that is

$$\phi(x) = \sum_{\mathbf{k}} \Big[\,a_{\mathbf{k}}\,u_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger}\,u_{\mathbf{k}}^{*}(x)\,\Big]$$

$$\Pi(x) = \sum_{\mathbf{k}} \mathrm{i}\omega_{\mathbf{k}}\Big[\,a_{\mathbf{k}}^{\dagger}\,u_{\mathbf{k}}^{*}(x) - a_{\mathbf{k}}\,u_{\mathbf{k}}(x)\,\Big] \tag{3.42}$$

$$[\,\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})\,] = 0 = [\,\Pi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,]$$

$$[\,\phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\,] \;=\; \mathrm{i}\,\delta\,(\mathbf{x}-\mathbf{y}) \tag{3.43}$$

This means that the classical expressions (3.31) and (3.33) of the energy and momentum of the free real scalar field will turn into the quantum operator expressions

$$
\begin{aligned}
P_0 &= \sum_{\mathbf{k}} \tfrac{1}{2}\left(\widehat{P}_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 \widehat{Q}_{\mathbf{k}}^2\right) \\
&= \sum_{\mathbf{k}} \tfrac{1}{2}\,\omega_{\mathbf{k}}\left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\right) \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta(\mathbf{0}) \sum_{\mathbf{k}} \tfrac{1}{2}\,\omega_{\mathbf{k}} \qquad (3.44) \\
\mathbf{P} &= \sum_{\mathbf{k}} \tfrac{1}{2}\,\mathbf{k}\left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\right) \\
&= \sum_{\mathbf{k}} \mathbf{k}\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \qquad (3.45)
\end{aligned}
$$

It is then easy to check that the definition of the conjugate momentum field $\Pi(x)$ and the Klein-Gordon field equation can be recast into the canonical Heisenberg form

$$
\dot{\phi}(x) = \frac{1}{\mathrm{i}\hbar}\,[\,\phi(x)\,,\,P_0\,] = \Pi(x)
$$

$$
\dot{\Pi}(x) = \frac{1}{\mathrm{i}\hbar}\,[\,\Pi(x)\,,\,P_0\,] = (\nabla^2 + m^2)\,\phi(x)
$$

It is worthwhile to spend some few words concerning the divergent quantity

$$
\begin{aligned}
cU_0 &= \delta(\mathbf{0}) \sum_{\mathbf{k}} \tfrac{1}{2}\,\hbar\omega_{\mathbf{k}} \equiv V \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\frac{1}{2}\,\hbar\omega_{\mathbf{k}} \\
&= V\hbar c \int_0^K \mathrm{d}k\,\frac{k^2}{4\pi^2}\,\sqrt{k^2 + m^2c^2/\hbar^2} \qquad (3.46)
\end{aligned}
$$

where V is the volume of a very large box and $\hbar K \gg mc$ is a very large wavenumber. The latter is called the *vacuum energy or zero–point energy of the real scalar field.* Since we know that a free real scalar field is dynamically equivalent to an infinite (continuous) set of linear oscillators, we can roughly understand the divergent quantity U_0 to be generated by summing up the quantum fluctuations of the canonical pair of operators $\phi(t,\mathbf{x})$ and $\Pi(t,\mathbf{x})$, *alias* $\widehat{Q}_{\mathbf{k}}$ and $\widehat{P}_{\mathbf{k}}$ ($\mathbf{k} \in \mathbb{R}^3$) at each point $\mathbf{x} \in V$ of a very large box in the three dimensional space.

It turns out to be more appropriate to discuss the *vacuum energy density*, *i.e.* the regularized vacuum energy *per* unit volume of the quantum real scalar field. The *vacuum state vector* or *vacuum state amplitude* of the actual quantum mechanical system under investigation is nothing but the ground state of the system and physically corresponds to the absence of field quanta, *i.e.* spinless massive particles of rest mass m. It is defined by

$$a_{\mathbf{k}}\,|\,0\,\rangle = 0 = \langle\,0\,|\,a_{\mathbf{k}}^{\dagger} \qquad \forall\,\mathbf{k}\in\mathbb{R}^3 \tag{3.47}$$

so that, after setting

$$\frac{c}{V}\,\langle\,0\,|\,P_0\,|\,0\,\rangle \equiv \langle\,\rho\,\rangle\,c^2 = \frac{\hbar c}{4\pi^2}\int_0^K k^2\,\mathrm{d}\,k\;\sqrt{k^2+(mc/\hbar)^2}$$

together with $\xi \equiv \hbar K/mc$, then we have[2]

$$\begin{aligned}
\langle\,\rho\,\rangle &= \frac{m^4c^3}{4\pi^2\hbar^3}\int_0^{\xi} x^2\sqrt{1+x^2}\;\mathrm{d}\,x \\
&= \frac{m^4c^3}{16\pi^2\hbar^3}\left[\,\xi(1+\xi^2)^{3/2}-\frac{1}{2}\,\xi(1+\xi^2)^{1/2}-\frac{1}{2}\,\ln\left(\xi+\sqrt{1+\xi^2}\right)\right] \\
&= \frac{\hbar K^4}{16\pi^2 c}+\frac{m^2cK^2}{16\pi^2\hbar}-\frac{m^4c^3}{32\pi^2\hbar^3}\left[\ln\left(\frac{\hbar K}{mc}\right)-\frac{1}{4}+\ln 2+O\left(\frac{mc}{\hbar K}\right)^2\right] \\
&\approx \frac{K^4}{16\pi^2}\left(\frac{\hbar}{c}\right) \qquad\quad (\hbar K \gg mc)
\end{aligned} \tag{3.48}$$

If we trust in general relativity and in quantum field theory up to the Planck scale

$$\begin{aligned}
M_P &= \sqrt{\hbar c/G_N} = 1.220\,90(9)\times 10^{\,19}\ \mathrm{GeV/c^2} \\
&= 2.176\,45(16)\times 10^{\,-11}\ \mathrm{g}
\end{aligned}$$

where G_N is the newtonian gravitational constant

$$G_N = 6.674\,2(10)\times 10^{\,-8}\ \mathrm{cm^3\ g^{\,-1}\ s^{\,-2}}$$

then we might take $K\ \simeq\ cM_P/\hbar$ which eventually gives

$$\langle\,\rho\,\rangle \approx \frac{M_P}{16\pi^2\,\ell_P^3} \approx 2\times 10^{\,115}\ (\,\mathrm{GeV}/c^2\,)\,\mathrm{cm}^{\,-\,3} \tag{3.49}$$

in which $\ell_P = \sqrt{\hbar G_N/c^3} = 1.61624(8)\times 10^{-33}$ cm denotes the Planck length. This is a truly enormously large value. So it is no surprise that Paul Adrien Maurice Dirac soon suggested that this zero–point energy must be simply discarded, as it turns out to be irrelevant for any laboratory experiment in which solely energy differences are indeed observable. However, Wolfgang

[2]See for example [22] eq. (**2.272** 2.) p.105

Pauli soon afterwards recognized that this vacuum energy surely couples to Einstein's gravity and it would then give rise to a large cosmological constant, so large that the size of the universe could not even reach the earth–moon distance. On the contrary, the present day observed value of the so called *dark energy density* of the universe is

$$\rho_\Lambda = \Omega_\Lambda\,\rho_c = \frac{3H_0^2}{8\pi G_N}\,\Omega_\Lambda = 1.05368(11)\times 10^{-5}\,\Omega_\Lambda^3\ (\,\mathrm{GeV}/c^2\,)\,\mathrm{cm}^{-3}$$

where $\Omega_\Lambda = 0.73(3)$ is the dark energy density fraction, $H_0 = 100\,\Omega_\Lambda$ Km s^{-1} Mpc^{-1} is the present day Hubble expansion rate. This leads to the cosmological constant value

$$\Lambda \;=\; 3H_0^2/c^2 = 3.50\times 10^{-52}\,\Omega_\Lambda^2\ \mathrm{m}^{-2} \tag{3.50}$$

which is extremely small but nonvanishing [3]. This eventually means that the dark energy density and the vacuum energy density of all the fundamental quantum fields in the universe differ by nearly 120 orders of magnitude : this is *the cosmological constant puzzle*, the solution of which is still unknown.

Leaving aside this intringuing puzzle, we now turn back to the realm of galileian laboratory experiments and endorse the Dirac's point of view. To this concern, we shall introduce the useful concept of an operator written in *normal form* as well as the concept of the *normal product* of operators [8]. The normal form of an operator involving products of creation and annihilation operators is said to be the form in which in each term all the creation operators are written to the left of all the annihilation operators. We consider an example. We write down in normal form the product of the two operators

$$\begin{aligned}
F(x)\,G(y) \;&\equiv\; \sum_{\mathbf{k}}\Big\{F_{\mathbf{k}}^*(x)\,a_{\mathbf{k}}^\dagger + F_{\mathbf{k}}(x)\,a_{\mathbf{k}}\Big\} \\
&\times\; \sum_{\mathbf{h}}\Big\{G_{\mathbf{h}}^*(y)\,a_{\mathbf{h}}^\dagger + G_{\mathbf{h}}(y)\,a_{\mathbf{h}}\Big\} \\
&=\; \sum_{\mathbf{k}}\sum_{\mathbf{h}}\ F_{\mathbf{k}}^*(x)\,G_{\mathbf{h}}^*(y)\,a_{\mathbf{k}}^\dagger\,a_{\mathbf{h}}^\dagger \;+\; \text{h.c.} \\
&+\; \sum_{\mathbf{k}}\sum_{\mathbf{h}}\Big(F_{\mathbf{k}}^*(x)\,G_{\mathbf{h}}(y) + F_{\mathbf{h}}(x)\,G_{\mathbf{k}}^*(y)\Big)\,a_{\mathbf{k}}^\dagger\,a_{\mathbf{h}} \\
&+\; \sum_{\mathbf{k}} F_{\mathbf{k}}(x)\,G_{\mathbf{k}}^*(y)
\end{aligned} \tag{3.51}$$

[3]This tiny but non–vanishing measured value of the cosmological constant does actually constitute a striking phenomenological evidence against supersymmetry.

The sum of terms **not involving** any ordinary c-number functions is called the normal product of the original operators. The normal product may also be defined as the original product reduced to its normal form with all the commutator functions being taken equal to zero in the process of reduction. The normal product of the operators $F(x)$ and $G(y)$ is denoted by the symbol

$$: F(x)\,G(y) : \stackrel{\text{def}}{=} F(x)\,G(y) - \sum_{\mathbf{k}} F_{\mathbf{k}}(x)\,G^*_{\mathbf{k}}(y)$$

We now agree by definition to express all dynamical variables which depend quadratically upon operators with the same arguments, such as the energy, momentum and angular momentum of the radiation fields, in the form of normal products. For example, we shall write the energy momentum four vector quantum operator

$$\begin{aligned} P_0 &= \frac{1}{2}\int \mathrm{d}\mathbf{x}\ : \Big\{\Pi^2(x) - \phi(x)\,\dot{\Pi}(x)\Big\} : \\ &= \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \qquad (3.52) \\ \mathbf{P} &= \int \mathrm{d}\mathbf{x}\ : \Pi(x)\boldsymbol{\nabla}\phi(x) : \\ &= \sum_{\mathbf{k}} \hbar\mathbf{k}\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \qquad (3.53) \end{aligned}$$

Now, if we keep the definition of the vacuum state to be given by (3.47) it follows that the expectation values of all the dynamical variables vanish for the vacuum state, *e.g.* $\langle P_\mu \rangle \equiv 0$. By this method we exclude from the theory at the outset the so called zero-point quantities of the type of the zero-point energy, which usually arise in the process of the quantization of the field theories and turn out to be, strictly speaking, mathematically ill-defined divergent quantities.

3.3 The Fock Space

The quantum theory of the free real scalar field naturally gives rise to the concept of spinless neutral massive *particle*. As a matter of fact, it appears to be clear that the nonnegative operators $a_{\mathbf{k}}^{\dagger}\, a_{\mathbf{k}}$ do possess integer eigenvalues $N_{\mathbf{k}} = 0, 1, 2, \ldots$ which are interpreted as the *numbers of particles* of a given wave number $\mathbf{k}$ and a given energy $\omega_{\mathbf{k}}$. The hamiltonian operator turns out to be positive definite and we can readily derive, from the canonical commutators (3.43), the eigenvalues and the common eigenstates of the energy momentum commuting operators P_μ : namely,

$$E\left(\{N_{\mathbf{k}}\}\right) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, N_{\mathbf{k}} \qquad N_{\mathbf{k}} = 0, 1, 2, \ldots \tag{3.54}$$

$$\mathbf{P}\left(\{N_{\mathbf{k}}\}\right) = \sum_{\mathbf{k}} \mathbf{k}\, N_{\mathbf{k}} \qquad N_{\mathbf{k}} = 0, 1, 2, \ldots \tag{3.55}$$

The setting up of the eigenstates leads to the well known construction of the so called *Fock space* :

Vladimir Alexandrovich Fock
Sankt Petersburg 22.12.1898 – 27.12.1974
Konfigurationsraum und zweite quantelung
Zeitschrift der Physik A **75**, 622–647 (1932).

Actually, in order to describe $N-$particle states, consider any set of particle numbers $\{N_{\mathbf{k}} = 0, 1, 2, \ldots \,|\, \mathbf{k} \in \mathbb{R}^3\}$ such that $\sum_{\mathbf{k}} N_{\mathbf{k}} = N$. Then the generic N–particle normalized energy momentum eigenstate can be written as

$$|\{N_{\mathbf{k}}\}\rangle_N \;\equiv\; \prod_{\mathbf{k}} (\, N_{\mathbf{k}}\,!\,)^{-1/2}\, a_{\mathbf{k}}^{\dagger\, N_{\mathbf{k}}}\, |\, 0\,\rangle$$

which satisfies

$$P_0\ |\{N_{\mathbf{k}}\}\rangle_N = E_N(\{N_{\mathbf{k}}\})\ |\{N_{\mathbf{k}}\}\rangle_N$$
$${}_M\,\langle \mathbf{h}\,|\,\mathbf{k}\rangle_N = \delta_{MN}\, \delta_{\mathbf{k}_1 \mathbf{h}_1}\, \delta_{\mathbf{k}_2 \mathbf{h}_2}\, \ldots\, \delta_{\mathbf{k}_N \mathbf{h}_N}$$

It is very important to gather that, owing to the commutation relation $[\, a_{\mathbf{h}}^{\dagger}\,,\, a_{\mathbf{k}}^{\dagger}\,] = 0$, by its very construction any N–particle state is completely symmetric under the exchange of any wave numbers, *i.e.* the scalar spinless particles obey the *Bose-Einstein statistics.*

In particular, the 1-particle energy momentum eigenstates are given by $|\,\mathbf{k}\,\rangle = a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle$ and satisfy

$$P_\mu\,|\,\mathbf{k}\,\rangle \;=\; k_\mu\,|\,\mathbf{k}\,\rangle \qquad k_0 = \omega_{\mathbf{k}}$$

The 1-particle wave functions in the coordinate representation, for a given wave number, are defined in terms of the matrix elements of the field operator (3.42) and read

$$\begin{aligned} u_{\mathbf{k}}\,(x) &\equiv \langle\, 0\,|\,\phi(x)\,|\,\mathbf{k}\,\rangle \\ &= [\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{-1/2}\,\exp\{-i\,k\cdot x\} \qquad k_0 = \omega_{\mathbf{k}} \end{aligned} \tag{3.56}$$

Notice that they turn out to be normalized in such a way to satisfy the orthonormality and closure relations

$$(\,u_{\mathbf{k}}\,,\,u_{\mathbf{h}}\,) \equiv \int d\mathbf{x}\; u^*_{\mathbf{k}}\,(x)\,i\,\overleftrightarrow{\partial}_0\,u_{\mathbf{h}}\,(x) \;=\; \delta\,(\mathbf{k}-\mathbf{h}) \tag{3.57}$$

$$\sum_{\mathbf{k}} \; u^*_{\mathbf{k}}\,(y)\,u_{\mathbf{k}}\,(x) \;\equiv\; \frac{1}{i}\,D^{\,(-)}\,(x-y) \tag{3.58}$$

$$D^{(\pm)}\,(x) = \frac{\mp i}{(2\pi)^3}\int \mathrm{d}k\;\delta\left(k^2-m^2\right)\,\theta\,(k_0)\,\exp\{\pm\,i\,k\cdot x\} \tag{3.59}$$

The 1-particle wave functions (3.56) satisfy by construction the Klein-Gordon wave equation

$$\left(\Box + m^2\right)\,u_{\mathbf{k}}\,(x) = 0 \qquad \forall\,\mathbf{k}\in\mathbb{R}^3$$

and do thereby represent a complete orthonormal base, with respect to the scalar product, in the 1-particle Hilbert space

$$\mathcal{H}_1 = \overline{\mathrm{V}}_1 \qquad \mathrm{V}_1 = \{\,|\,\mathbf{k}\,\rangle = a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle \quad \mathbf{k}\in\mathbb{R}^3\}$$

More precisely, eq. (3.56) does explicitly realize the isomorphism between the Fock space representation $\mathcal{H}_1$ of the space of 1-particle states and the spacetime coordinate representation $L^2(\mathbb{R}^3)$ of the 1-particle wave functions with respect to (3.57).

It turns out that the previously introduced 1-particle energy momentum eigenstates $|\,\mathbf{k}\,\rangle = a^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle$ are improper and have to be normalized, in the sense of tempered distribution according to

$$\langle\,\mathbf{h}\,|\,\mathbf{k}\,\rangle = \delta\,(\mathbf{h}-\mathbf{k}\,) \tag{3.60}$$

Moreover they satisfy the closure or completeness relation in the 1-particle Hilbert space $\mathcal{H}_1$ that reads

$$\sum_{\mathbf{k}}\;|\,\mathbf{k}\,\rangle\langle\,\mathbf{k}\,| = \mathbb{I}_1 \tag{3.61}$$

It is important to remark that these orthonormality and closure relationships, as well as the normal mode decompositions (3.42), are not Lorentz covariant.

As a consequence, the insofar developed quantization procedure for a real scalar field is set up in a particular class of inertial reference frames connected by spatial rotations belonging to the group $SO(3)$.

The $N-$particle wave functions in the coordinates representation can be readily obtained in the completely symmetric form

$$\begin{aligned} u_N(x_1, x_2, \ldots, x_N) &= \prod_{j=1}^{N} (N_{\mathbf{k}_j}!)^{-1/2}\, u_{\mathbf{k}_j}(x_j) \\ &\equiv \langle x_1 x_2 \ldots x_N \,|\, \mathbf{k}_1 \mathbf{k}_2 \ldots \mathbf{k}_N \rangle \end{aligned} \tag{3.62}$$

Furthermore, we can write the generic normalized element of the N–particle completely symmetric Hilbert space – the closure of the symmetric product of 1–particle Hilbert spaces

$$\mathcal{H}_N \equiv \overline{\mathrm{V}}_N \qquad \mathrm{V}_N = \{\underbrace{\mathcal{H}_1 \overset{\mathrm{s}}{\otimes} \mathcal{H}_1 \overset{\mathrm{s}}{\otimes} \ldots \overset{\mathrm{s}}{\otimes} \mathcal{H}_1}_{\mathrm{N\ times}}\} = \mathcal{H}_1^{\overset{\mathrm{s}}{\otimes} n}$$

in the form

$$|\varphi_N\rangle \equiv \sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\cdots\sum_{\mathbf{k}_N} C(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)\ |\{N_{\mathbf{k}}\}\rangle_N$$

with

$$\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\cdots\sum_{\mathbf{k}_N} |\, C(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)\,|^2 = 1$$

To end up, we are now able to write the generic normalized element of the Fock space of the spinless neutral scalar particle states

$$\mathcal{F} \ \equiv\ \mathbf{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus\ \ldots\ \oplus \mathcal{H}_N\ \oplus\ \ldots\ =\ \bigoplus_{n=1}^{\infty} \mathcal{H}_1^{\overset{\mathrm{s}}{\otimes} n}$$

in the form

$$|\,\Phi\,\rangle \ =\ \sum_{N=0}^{\infty} C_N\, |\,\varphi_N\,\rangle \qquad\qquad \sum_{N=0}^{\infty} |\, C_N\,|^2 = 1$$

which summarizes the setting up of the Fock space of the particle states, the structure of which is characterized by the canonical quantum algebra (3.43) and the energy momentum operators (3.53). From the above construction of the Fock space of the states of a free real scalar field, it appears quite evident that all the quantum states can be generated by linear combinations of repeated applications of the creation operators on the vacuum state. This property is known as the *cyclicity of the vacuum state.*

We end up this section by discussing the behaviour of the N-particle states and of the field operators under Lorentz transformations. First of all, we recall that the above introduced 1-particle states of the basis $\{|\,\mathbf{k}\,\rangle = a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle\}$ in the Hilbert space $\mathcal{H}_1$ do not satisfy covariant orthonormality and completeness relations. To remedy this, consider first the completeness relation and the trivial identity

$$\begin{aligned}
\sum_{\mathbf{k}} |\,\mathbf{k}\,\rangle\langle\,\mathbf{k}\,| \quad &= \quad \int \mathrm{d}\mathbf{k}\ a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle\langle\,0\,|\,a_{\mathbf{k}} \\
&= \quad \int \frac{d\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\ [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{\frac{1}{2}}\ a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle\langle\,0\,|\,a_{\mathbf{k}}\,[\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{\frac{1}{2}} \\
&\stackrel{\mathrm{def}}{=} \quad \int \mathrm{D}k\ a^{\dagger}(k)\,|\,0\,\rangle\langle\,0\,|\,a(k) \equiv \int \mathrm{D}k\ |\,k\,\rangle\langle\,k\,| = \mathbf{I}_1
\end{aligned}$$

whence we get

$$\int \mathrm{D}k \quad \stackrel{\mathrm{def}}{=} \quad \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}} = \frac{1}{(2\pi)^3}\int \mathrm{d}k\ \theta(k_0)\,\delta\left(k^2 - m^2\right) \tag{3.63}$$

$$|\,k\,\rangle \quad \stackrel{\mathrm{def}}{=} \quad [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{\frac{1}{2}}\ a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle = a^{\dagger}(k)\,|\,0\,\rangle \tag{3.64}$$

The Lorentz invariant completeness relation for the 1-particle states can also be written in the two equivalent forms

$$\sum_{\mathbf{k}} |\,\mathbf{k}\,\rangle\,\langle\,\mathbf{k}\,| \; = \; \mathbf{I}_1 \; = \; \int \mathrm{D}k\ |\,k\,\rangle\langle\,k\,| \tag{3.65}$$

Now it is clear that the 1-particle states of the new basis, which will be named *covariant 1-particle states*,

$$\{\,|\,k\,\rangle = [\,(2\pi)^3\,2\omega_{\mathbf{k}}\,]^{\frac{1}{2}}\ a_{\mathbf{k}}^{\dagger}\,|\,0\,\rangle\,|\ \mathbf{k}\in\mathbb{R}^3\} \tag{3.66}$$

fulfill *manifestly Lorentz covariant* orthonormality relations

$$\langle\,k'\,|\,k\,\rangle \; = \; (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta\,(\mathbf{k}-\mathbf{k}') \tag{3.67}$$

Furthermore we can write

$$\phi(x) \; \equiv \; \int \mathrm{D}k\ \left[\,a\,(k)\,e^{-ikx} + a^{\dagger}(k)\,e^{ikx}\,\right] \tag{3.68}$$

$$\Pi(x) \; \equiv \; \int \mathrm{D}k\ k_0\left[\, -\ i\,a\,(k)\,e^{-ikx} + i\,a^{\dagger}(k)\,e^{ikx}\,\right] \tag{3.69}$$

so that the 1-particle wave functions in the coordinate representation, which correspond to the 1-particle state $|\,k\,\rangle$ and are still defined in terms of the matrix elements of the field operator (3.42), will coincide with the plane waves

$$u_k\,(x) \;\equiv\; \langle\, 0\,|\, \phi\,(x)\,|\, k\,\rangle = \exp\{-i\,k\cdot x\} \qquad (\,k_0 = \omega_{\mathbf{k}}\,) \tag{3.70}$$

which are normalized in such a way to satisfy

$$\int \mathrm{d}\mathbf{x}\; u^*_k\,(x)\,i\,\overleftrightarrow{\partial}_0\,u_h\,(x) \;=\; 2\omega_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{h}) \tag{3.71}$$

Now it becomes clear that to each element of the restricted Lorentz group, which is univocally specified by the six canonical coordinates $\omega^{\,\mu\nu} = (\boldsymbol{\alpha},\boldsymbol{\eta})$, there will correspond a *unitary operator* $U(\omega)\,:\,\mathcal{H}_1\;\rightarrow\;\mathcal{H}_1$ so that

$$U(\omega)\,|\,k\,\rangle = |\,\Lambda\,k\,\rangle \qquad \langle\,k\,|\,U^{\,\dagger}(\omega) = \langle\,\Lambda\,k\,| \tag{3.72}$$

By virtue of the covariant orthonormality relation (3.67) we can write

$$\langle\,\Lambda\,h\,|\,\Lambda\,k\,\rangle = \langle\,h\,|\,k\,\rangle = \langle\,h\,|\,U^{\,\dagger}(\omega)\,U(\omega)\,k\,\rangle = \delta\,(\mathbf{h}-\mathbf{k}\,)\,(2\pi)^3\,2\omega_{\mathbf{k}} \tag{3.73}$$

where we obviously understand *e.g.*

$$\begin{aligned} &|\,\Lambda\,k\,\rangle = |\,k'\,\rangle = [\,(2\pi)^3\,2\,k'_0\,]^{\frac{1}{2}}\,a^{\dagger}_{\mathbf{k}'}\,|\,0\,\rangle = a^{\dagger}(\Lambda\,k)\,|\,0\,\rangle \\ &k'_\mu = \Lambda_\mu^{\;\;\nu}\,k_\nu \qquad k_0 = \omega_{\mathbf{k}} \\ &g^{\,\mu\nu}\,k'_\mu\,k'_\nu = k^2 = m^2 \end{aligned} \tag{3.74}$$

In turn, under a Lorentz transformation the creation annihilation operator undergo the change

$$U(\omega)\,a^{\dagger}(k)\,U^{\,\dagger}(\omega) = a^{\dagger}(\Lambda\,k) \qquad U(\omega)\,a\,(k)\,U^{\,\dagger}(\omega) = a\,(\Lambda\,k) \tag{3.75}$$

under the assumption that the vacuum is invariant *i.e.* $U(\omega)\,|0\rangle = |\,0\,\rangle$.

As a consequence we immediately obtain the transformation law of the quantized real scalar field under passive Lorentz transformations : namely,

$$\begin{aligned} \phi'(x) &\stackrel{\text{def}}{=} U(\omega)\,\phi\,(x)\,U^{\,\dagger}(\omega) \\ &= \int \mathrm{D}\,k\,\left[\,a\,(\Lambda\,k)\,e^{\,-ikx} + a^{\dagger}(\Lambda\,k)\,e^{\,ikx}\,\right] \\ &= \int \mathrm{D}\,k'\;a(k')\,\exp\left\{\,-\mathrm{i}\,(\Lambda^{-1}k')_\mu\,x^\mu\right\} + \text{h. c.} \\ &= \phi\,(\Lambda\,x) \end{aligned} \tag{3.76}$$

where I have made use of the relation in matrix notation

$$\begin{aligned}(\Lambda^{-1}k')_\mu\, x^\mu &= (\Lambda^{-1}k')^\top g\, x = k'^\top (\Lambda^{-1})^\top g\, x \\ &= k'^\top (\, g\,\Lambda\, g)\, g\, x = k'^\top g\,\Lambda\, x = k'_\mu\, \Lambda^\mu{}_\nu\, x^\nu\end{aligned}$$

Let us now collect the ten conserved dynamical quantities related to the quantized real scalar field : namely,

$$\begin{aligned}P_0 &= \frac{1}{2}\int \mathrm{d}\mathbf{x}\ :\Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) : \\ P_k &= \int \mathrm{d}\mathbf{x}\ :\Pi(x)\nabla_k\phi(x) : \\ L_{jk} &= \int \mathrm{d}\mathbf{x}\ : x_j\,\Pi(x)\nabla_k\phi(x) - x_k\,\Pi(x)\nabla_j\phi(x) : \\ L_{0k} &= x_0\, P_k \\ &- \frac{1}{2}\int \mathrm{d}\mathbf{x}\, x_k : \Pi^2(x) + \boldsymbol{\nabla}\phi(x)\cdot\boldsymbol{\nabla}\phi(x) + m^2\phi^2(x) :\end{aligned}$$

Owing to the self-adjointness of the field operators $\phi(x) = \phi^\dagger(x)$ and $\Pi(x) = \Pi^\dagger(x)$ all the above ten conserved dynamical quantities turn out to be self–adjoint operators corresponding to physical observables. For example

$$\begin{aligned}\mathbf{P}^\dagger &= \int \mathrm{d}\mathbf{x}\ :[\,\boldsymbol{\nabla}\phi^\dagger(x)\,]\,\Pi^\dagger(x) : \\ &= \int \mathrm{d}\mathbf{x}\ :[\,\boldsymbol{\nabla}\phi(x)\,]\,\Pi(x) : \\ &= \int \mathrm{d}\mathbf{x}\ :\Pi(x)\boldsymbol{\nabla}\phi(x) : \end{aligned} \tag{3.77}$$

thanks to normal ordering. It appears thereby evident that the ten selfadjoint operators $(P_\mu\,,\,L_{\rho\sigma})$ acting on the Fock space are the generators of a unitary infinite dimensional representation of the Poincaré group on $\mathcal{F}$. From the canonical commutation relations (3.43) it is immediate to show that

$$[\,\phi(x)\,,\,P_\mu\,] = \mathrm{i}\,\partial_\mu\,\phi(x) \tag{3.78}$$

$$[\,\phi(x)\,,\,L_{\mu\nu}\,] = \mathrm{i}\,x_\mu\,\partial_\nu\,\phi(x) - \mathrm{i}\,x_\nu\,\partial_\mu\,\phi(x) \tag{3.79}$$

By direct inspection, it is straightforward to verify that, using the canonical commutation relations (3.43) and the normal ordering prescription, the self-adjoint operators $(P_\mu\,,\,L_{\rho\sigma})$ do actually fulfill the Poincaré Lie algebra (1.40) so that, in any neighbourhood of the unit element $\omega^{\,\rho\sigma} = 0 = a^\mu$, we can safely write

$$U\,:\,\mathcal{F}\to\mathcal{F} \qquad U(\omega,a) = \exp\left\{\mathrm{i}\,a^\mu\,P_\mu - \frac{\mathrm{i}}{2}\,\omega^{\,\rho\sigma}L_{\rho\sigma}\right\}$$

This is the way how to realize the unitary irreducible representation of the Poincaré group with mass m and spin zero from the quantization of the real scalar free field. In fact, for an infinitesimal Poincaré transformation we have

$$\begin{aligned}\phi'(x) &\equiv U(\delta\omega,\delta a)\,\phi(x)\,U^{\dagger}(\delta\omega,\delta a)\\ &= \phi(x)+\mathrm{i}\,\epsilon^{\mu}\,[\,P_{\mu}\,,\,\phi(x)\,]-\tfrac{\mathrm{i}}{2}\,\epsilon^{\,\rho\sigma}\,[\,L_{\rho\sigma}\,,\,\phi(x)\,]\\ &= \phi(x)+(\,\epsilon^{\mu}+\epsilon^{\,\mu\nu}\,x_{\nu}\,)\,\partial_{\mu}\,\phi(x)\\ &= \phi(x+\delta x)\end{aligned} \tag{3.80}$$

in which we have denoted by $\delta a^{\mu}=\epsilon^{\mu}$ and $\delta\omega^{\,\rho\sigma}=\epsilon^{\,\rho\sigma}$ the infinitesimal parameters.

3.4 Special Distributions

We have already met the positive and negative frequency scalar distributions

$$D^{(\pm)}(x) = \pm\,\frac{1}{\mathrm{i}}\int\frac{\mathrm{d}\,k}{(2\pi)^3}\;\exp\{\pm\,\mathrm{i}\,k\cdot x\}\;\delta\left(k^2-m^2\right)\,\theta\,(k_0)$$

The latter ones are characterized by

$$\langle\, 0\,|\,\phi\,(x)\,\phi\,(y)\,|\,0\,\rangle = \frac{1}{\mathrm{i}}\,D^{\,(-)}\,(x-y)$$

$$D^{\,(+)}\,(x-y) = -\,D^{\,(-)}\,(y-x)$$

$$[\,D^{\,(\pm)}\,(x)\,]^* \;=\; D^{\,(\mp)}\,(x)$$

From the normal mode expansion (3.42) and the canonical commutation relations (3.43) we obtain the commutator between two real scalar free field operator at arbitrary points, which is known as the *Pauli-Jordan* distribution

$$[\,\phi(x)\,,\,\phi(y)\,] \equiv \frac{1}{\mathrm{i}}\,D\,(x-y) \tag{3.81}$$

where

$$\begin{aligned} D\,(x) \;&\stackrel{\text{def}}{=}\; \mathrm{i}\int\frac{\mathrm{d}\,k}{(2\pi)^3}\;\exp\{-\,\mathrm{i}\,k\cdot x\}\,\delta\left(k^2-m^2\right)\,\mathrm{sgn}\,(k_0)\\ &\equiv\; D^{(-)}\,(x) \;+\; D^{(+)}\,(x) \end{aligned} \tag{3.82}$$

where the sign distribution is defined to be

$$\mathrm{sgn}\,(k_0) = \theta(k_0) - \theta(-k_0) = \begin{cases} +1 & \text{for } k_0 > 0\\ -1 & \text{for } k_0 < 0\end{cases}$$

The Pauli-Jordan distribution is a Poincaré invariant solution of the Klein-Gordon wave equation

$$\left(\Box_x + m^2\right)\,D\,(x-y) = 0$$

with the initial conditions

$$\lim_{x_0\,\to\,y_0}\;D\,(x-y) = 0 \qquad\qquad \lim_{x_0\,\to\,y_0}\;\frac{\partial}{\partial x_0}\;D\,(x-y) \;=\; \delta\,(\mathbf{x}-\mathbf{y})$$

in such a manner that

$$\lim_{x_0\,\to\,y_0}\;[\,\Pi\,(x)\,,\,\phi\,(y)\,] = \lim_{x_0\,\to\,y_0}\;\frac{1}{\mathrm{i}}\,\partial_0\,D\,(x-y) = \;-\,\mathrm{i}\,\delta\,(\mathbf{x}-\mathbf{y})$$

in accordance with the canonical equal-time commutation relations.

The Pauli-Jordan distribution is real

$$D^*(x) = D(x)$$

and enjoys as well the very important property of vanishing for spacelike separations, that is

$$D(x-y) = 0 \qquad \text{for} \quad (x_0 - y_0)^2 < (\mathbf{x} - \mathbf{y})^2 \tag{3.83}$$

The above feature is known as the *microcausality property*.

A further very important distribution related to causality is the *causal Green function* or *Feynman propagator*. It is defined as follows:

$$\begin{aligned} D_F(x-y) &= \begin{cases} \langle 0 | \phi(x)\,\phi(y) | 0 \rangle & \text{for } x_0 > y_0 \\ \langle 0 | \phi(y)\,\phi(x) | 0 \rangle & \text{for } x_0 < y_0 \end{cases} \\ &= \theta(x_0 - y_0)\,\langle 0 | \phi(x)\,\phi(y) | 0 \rangle \\ &+ \theta(y_0 - x_0)\,\langle 0 | \phi(y)\,\phi(x) | 0 \rangle \\ &\equiv \langle 0 | T\,\phi(x)\,\phi(y) | 0 \rangle \end{aligned} \tag{3.84}$$

the last line just defining the *chronological product* of operators in terms of the *time ordering* symbol T that prescribes the place for the operators that follow in the order with the latest to the left. It is easy to check, by applying the Klein-Gordon differential operator $\Box_x + m^2$ to the Feynman propagator and taking (3.43) into account, that the causal Green function is a solution of the inhomogeneous equation

$$(\Box_x + m^2)\,D_F(x-y) = -\,\mathrm{i}\,\delta(x-y)$$

so that its Fourier representation reads

$$D_F(x-y) = \frac{\mathrm{i}}{(2\pi)^4} \int \frac{\exp\{-\,\mathrm{i}\,k\cdot(x-y)\}}{k^2 - m^2 + \mathrm{i}\varepsilon}\,\mathrm{d}\,k \tag{3.85}$$

In fact, from the integral representation of the Heaviside step distribution

$$\theta\,(\pm\, z^0) \;=\; \pm \lim_{\varepsilon\,\to\,0^+} \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{p_0 \pm \mathrm{i}\,\varepsilon}\, \exp\{-\,\mathrm{i}\,p_0\, z^0\} \tag{3.86}$$

and from the normal mode expansion of the free real scalar field

$$\begin{aligned} \phi(x) &= \sum_{\mathbf{k}} \left[\, a_{\mathbf{k}}\, u_{\mathbf{k}}(x) + a^{\dagger}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(x) \,\right] \\ u_{\mathbf{k}}(x) &\equiv [\,2\omega_{\mathbf{k}}(2\pi)^3\,]^{-1/2}\, \exp\{-\,\mathrm{i}\,x_0\,\omega_{\mathbf{k}} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\} \end{aligned}$$

by making use of the canonical commutation relations (3.43) we can write

$$
\begin{aligned}
\langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle \;&=\; \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\;\exp\{\mathrm{i}\,(k_0-\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,\left[\,2\,\omega_{\mathbf{k}}\,(k_0-\mathrm{i}\,\varepsilon)\,\right]^{-1}\\
&\times\;\exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}\\
&+\;\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\exp\{-\,\mathrm{i}\,(k_0-\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,\left[\,2\,\omega_{\mathbf{k}}\,(k_0-\mathrm{i}\,\varepsilon)\,\right]^{-1}\\
&\times\;\exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}
\end{aligned}
$$

Changing the integration variable from k_0 to $k_0'=-\,k_0$ in the first integral of the right hand side of the previous equality we obtain

$$
\begin{aligned}
D_F(x-y)\;&=\;-\,\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\;\exp\{-\,\mathrm{i}\,(k_0+\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,\left[\,2\,\omega_{\mathbf{k}}\,(k_0+\mathrm{i}\,\varepsilon)\,\right]^{-1}\\
&\times\;\exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}\\
&+\;\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}\,k_0\;\exp\{-\,\mathrm{i}\,(k_0-\omega_{\mathbf{k}})\,(x_0-y_0)\}\\
&\times\;\sum_{\mathbf{k}}\,(2\pi)^{-3}\,\left[\,2\,\omega_{\mathbf{k}}\,(k_0-\mathrm{i}\,\varepsilon)\,\right]^{-1}\\
&\times\;\exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\}
\end{aligned}
$$

and a translation with respect to the k_0 integration variable yields

$$
\begin{aligned}
\langle\,T\,\phi(x)\,\phi(y)\,\rangle\;&=\;\frac{\mathrm{i}}{(2\pi)^4}\int\mathrm{d}\,k\;\;\exp\{-\,\mathrm{i}\,k\cdot(x-y)\}\,[\,2\,\omega_{\mathbf{k}}\,]^{-1}\\
&\times\;\left(\frac{1}{k_0-\omega_{\mathbf{k}}+\mathrm{i}\,\varepsilon}\;-\;\frac{1}{k_0+\omega_{\mathbf{k}}-\mathrm{i}\,\varepsilon}\right)\\
&=\;\int\frac{\mathrm{d}\,k}{(2\pi)^4}\;e^{-\,\mathrm{i}\,k\cdot(x-y)}\;\frac{\mathrm{i}}{k^2-m^2+i\varepsilon}
\end{aligned}
$$

which proofs the Fourier representation (3.85).

The latter one just involves the $(+\mathrm{i}\varepsilon)$ prescription in momentum space that corresponds to causality in coordinate space. As a matter of fact, if we define the creation $\phi^{(+)}(x)$ and destruction $\phi^{(-)}(x)$ parts of the free real scalar field operator according to

$$\phi^{(-)}(x) = \sum_{\mathbf{k}} a_{\mathbf{k}}\, u_{\mathbf{k}}(x) \tag{3.87}$$

$$\phi^{(+)}(x) = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \tag{3.88}$$

it turns out that we have

$$\begin{aligned}
\langle T\,\phi(x)\,\phi(y)\,\rangle_0 &= \theta(x_0 - y_0)\,\langle\,\phi^{(-)}(x)\,\phi^{(+)}(y)\,\rangle_0 \\
&+ \theta(y_0 - x_0)\,\langle\,\phi^{(-)}(y)\,\phi^{(+)}(x)\,\rangle_0 \\
&= \theta(x_0 - y_0)\,\frac{1}{\mathrm{i}}\,D^{(-)}(x-y) + \theta(y_0 - x_0)\,\frac{1}{\mathrm{i}}\,D^{(-)}(y-x) \\
&= \mathrm{i}\,\theta(y_0 - x_0)\,D^{(+)}(x-y) - \mathrm{i}\,\theta(x_0 - y_0)\,D^{(-)}(x-y)
\end{aligned} \tag{3.89}$$

which shows that, first, a particle is created out of the vacuum by the creation part of the free real scalar field operator and then, later, it is annihilated by the destruction part of the free real scalar field operator : the opposite never occurs, what precisely endorses the causality requirement in coordinate space.

3.4.1 Wick Rotation : Euclidean Formulation

The causal $+\mathrm{i}\varepsilon$ prescription in momentum space enjoys a crucial feature, the so called *Wick rotation* property. To see this, consider once again the Fourier representation (3.85) and change the energy integration variable as $k_0 = -\,\mathrm{i}\,k_4$ so that

$$D_F\,(x) \;=\; \frac{\mathrm{i}}{(2\pi)^4}\int_{-\infty}^{\infty} \mathrm{d}\,(-\,\mathrm{i}\,k_4)\sum_{\mathbf{k}} \frac{\exp\{-\,k_4 x_0 + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\}}{-k_4^2 - \mathbf{k}^2 - m^2} \tag{3.90}$$

where the $+\mathrm{i}\varepsilon$ prescription has been dropped since the denominator is now positive definite. If we further set $\mathrm{i}\,x_0 \equiv x_4$ then we finally get

$$\begin{aligned}
D_E\,(x_E\,) &= -\,D_F\,(-\,\mathrm{i}x_4, \mathbf{x}) \\
&= \frac{1}{(2\pi)^4}\int \mathrm{d}\,k_E\,\exp\{\mathrm{i}\,k_{E\mu}\,x_{E\mu}\}\,\left(k_E^2 + m^2\right)^{-1}
\end{aligned} \tag{3.91}$$

where we use the notation

$$x_E = x_{E\mu} = (x_k, x_4) \qquad k_E = k_{E\mu} = (k_j, k_4) \qquad (\,j, k = 1, 2, 3\,)$$
$$k_E \cdot x_E = k_{E\mu}\,x_{E\mu} = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4$$
$$\int \mathrm{d}\,k_E = \int_{-\infty}^{\infty} \mathrm{d}\,k_4 \sum_{\mathbf{k}}$$

The Wick rotation is nothing but a counterclockwise rotation of a $\pi/2$ angle in the complex energy plane that drives the real k_0 axis over the imaginary k_4 axis. The location of the poles of the Feynman propagator in the complex energy plane, that corresponds to the causal $+\mathrm{i}\varepsilon$ prescription, is such that no singularity crossing occurs owing to the Wick rotation. It is precisely this crucial aspect that encodes the causality requirement in momentum space.

In the configuration space we turn to the so called euclidean formulation, according to which the action and the lagrangian in the Minkowski spacetime are transformed into the purely imaginary euclidean action and Lagrange density. As a matter of fact, if we change the time integration variable according to $x_0 = -\,\mathrm{i}\,x_4$, then we readily obtain

$$\begin{aligned} S[\,\phi\,] \;&\mapsto\; S_E[\,\phi_E\,] \\ &= \frac{\mathrm{i}}{2}\int \mathrm{d}x_E\;\Big(\partial_\mu\phi_E(x_E)\,\partial_\mu\phi_E(x_E) + m^2\,\phi_E^2(x_E)\Big) \end{aligned} \tag{3.92}$$

with the euclidean indices always lower case so that

$$\partial_\mu\phi_E \equiv \frac{\partial\phi_E}{\partial x_{E\mu}} \qquad (\,\mu = 1,2,3,4\,)$$

If we assume the asymptotic behaviour for the euclidean scalar field

$$\lim_{x_E\to\infty}\;\phi_E(x_E)\,\sqrt{x_E^2} \;=\; 0 \tag{3.93}$$

then we can also write

$$\begin{aligned} S_E[\,\phi_E\,] \;&=\; \frac{\mathrm{i}}{2}\int \mathrm{d}x_E\;\phi_E(x_E)\,(-\,\partial_E^2 + m^2)\,\phi_E(x_E) \\ &=\; \frac{\mathrm{i}}{2}\int \mathrm{d}k_E\;\tilde\phi_E(k_E)(\,k_E^2 + m^2\,)\,\tilde\phi_E(-\,k_E) \end{aligned} \tag{3.94}$$

in which I have set by definition

$$\phi_E(x_E) \;\equiv\; \frac{1}{(2\pi)^4}\int \mathrm{d}k_E\;\tilde\phi_E(k_E)\;\exp\{\,\mathrm{i}\,k_{E\mu}\,x_{E\mu}\,\}$$

3.5 The Generating Functional

3.5.1 The Symanzik Functional Equation

Consider the vacuum expectation value

$$
\begin{aligned}
Z_0[\,J\,] \;&=\; \left\langle T \exp\left\{ \mathrm{i}\int \mathrm{d}x\ \phi(x)\,J(x)\right\}\right\rangle_0 \\
&\stackrel{\mathrm{def}}{=}\; \sum_{n=0}^{\infty}\frac{\mathrm{i}^{\,n}}{n!}\int \mathrm{d}x_1\,J(x_1)\cdots\int \mathrm{d}x_n\,J(x_n) \\
&\times\;\; \langle\,0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\,\rangle
\end{aligned}
\qquad (3.95)
$$

the suffix zero denoting the free field theory, where $J(x)$ are the so called classical external sources with the canonical dimensions $[\,J\,] = \mathrm{eV}^{\,3}$. The vacuum expectation values of the chronological ordered products of n free scalar field operators at different spacetime points are named the $n-$point Green functions of the (free) field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional : namely,

$$
\begin{aligned}
G_0^{(n)}(x_1,\cdots,x_n) \;&\equiv\; \langle\,0\,|\,T\,\phi\,(x_1)\cdots\phi\,(x_n)\,|\,0\,\rangle \\
&=\; (\,-\,\mathrm{i}\,)^{\,n}\,\delta^{\,(n)}\,Z_0\,[\,J\,]\,/\,\delta\,J(x_1)\cdots\delta\,J(x_n)\,\big\rfloor_{J=0}
\end{aligned}
\qquad (3.96)
$$

Taking one functional derivative of the generating functional (3.95) we find

$$
(\,-\,\mathrm{i}\,)\,\frac{\delta}{\delta\,J(x)}\;Z_0\,[\,J\,] = \left\langle T\,\phi(x)\,\exp\left\{\mathrm{i}\int \mathrm{d}y\ \phi(y)\,J(y)\right\}\right\rangle_0 \qquad (3.97)
$$

In order to evaluate the above quantity it is convenient to introduce the operator [11]

$$
E(t^{\,\prime}\,,\,t\,) \equiv T\,\exp\left\{\mathrm{i}\int_t^{t^{\,\prime}}\mathrm{d}y^0\int \mathrm{d}\mathbf{y}\ \phi(y)\,J(y)\right\} \qquad (3.98)
$$

so that we can write

$$
(\,-\,\mathrm{i}\,)\,\frac{\delta}{\delta\,J(x)}\;Z_0\,[\,J\,] = \langle\,0\,|\,E(\infty\,,\,x_0\,)\,\phi(x)\,E(x_0\,,\,-\,\infty\,)\,|\,0\,\rangle \qquad (3.99)
$$

Taking a derivative with respect to x_0 we find

$$
\begin{aligned}
&\frac{\partial}{\partial\,x_0}\,\langle\,0\,|\,E(\infty\,,\,x_0\,)\,\phi(x)\,E(x_0\,,\,-\,\infty\,)\,|\,0\,\rangle \;= \\
&\langle\,0\,|\,E(\infty\,,\,x_0\,)\,\Pi(x)\,E(x_0\,,\,-\,\infty\,)\,|\,0\,\rangle \;- \\
&\mathrm{i}\,\langle\,0\,|\,E(\infty\,,\,x_0\,)\int \mathrm{d}\mathbf{y}\,[\,\phi(x_0,\mathbf{y})\,,\,\phi(x_0,\mathbf{x})\,]\,J(x_0,\mathbf{y})\,E(x_0\,,\,-\,\infty\,)\,|\,0\,\rangle \\
&=\;\langle\,0\,|\,E(\infty\,,\,x_0\,)\,\Pi(x)\,E(x_0\,,\,-\,\infty\,)\,|\,0\,\rangle
\end{aligned}
\qquad (3.100)
$$

owing to microcausality. One more derivative evidently yields

$$\frac{\partial}{\partial x_0}\,\langle 0|\,E(\infty\,,\,x_0\,)\,\Pi(x)\,E(x_0\,,\,-\infty\,)\,|\,0\rangle \;=\;$$
$$\langle 0|\,E(\infty\,,\,x_0\,)\,\dot{\Pi}(x)\,E(x_0\,,\,-\infty\,)\,|\,0\rangle \;+\; J(x)\,Z_0[\,J\,]$$

whence we eventually obtain the functional differential equation for the free scalar field generating functional, that is

$$\Big[\,(\,\Box_x+m^2\,)\,\frac{\mathrm{i}\,\delta}{\delta J(x)}+J(x)\Big]\,Z_0[\,J\,] \;=\; 0 \tag{3.101}$$

where we used the fact that the free scalar field operator valued distribution does satisfy the Klein-Gordon wave equation. The above functional equation has been first obtained by

Kurt Symanzik
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and will thereby named the *Symanzik functional equation*. This functional differential equation (3.101) has a *unique solution that fulfills causality*

$$Z_0[\,J\,]=\exp\left\{-\int\mathrm{d}\,x\int\mathrm{d}\,y\;\tfrac12\,J(x)\,D_F\,(x-y)\,J(y)\right\} \tag{3.102}$$

as it can be seen by direct substitution.

The classical action for the real scalar field in the presence of an external source can be rewritten as

$$\begin{aligned} S_0[\,\phi\,,\,J\,] \;&=\; \int\mathrm{d}\,x\;\Big[\,\tfrac12\,g^{\mu\nu}\,\partial_\mu\phi(x)\partial_\nu\phi(x)-\tfrac12\,m^2\,\phi^{\,2}(x)+\phi(x)\,J(x)\Big] \\ &\doteq\; \int\mathrm{d}\,x\;\Big[\,-\,\tfrac12\,\phi(x)\,\big(\Box+m^2\big)\,\phi(x)+\phi(x)\,J(x)\Big] \\ &\equiv\; S_0[\,\phi\,]+\int\mathrm{d}\,x\;\phi(x)\,J(x) \end{aligned} \tag{3.103}$$

where the symbol $\doteq$ indicates that the total four divergence term

$$\tfrac12\,g^{\mu\nu}\int\mathrm{d}\,x\;\partial_\mu\Big(\phi(x)\partial_\nu\phi(x)\Big)$$

has been dropped as it does not contribute to the equations of motion

$$-\,\delta S_0[\,\phi\,]\,/\,\delta\,\phi\,(x)=(\,\Box+m^2\,)\,\phi(x)=J(x) \tag{3.104}$$

3.5.2 The Functional Integral

To reconstruct the solution (3.101) *via* Fourier methods we formally write $Z_0[\,J\,]$ as a *functional Fourier transform*

$$Z_0[\,J\,] = \int \mathfrak{D}\phi\ \widetilde{Z}_0[\,\phi\,]\ \exp\left\{\mathrm{i}\int \mathrm{d}x\ \phi(x)\,J(x)\right\} \tag{3.105}$$

where $\mathfrak{D}\phi$ formally denotes integration over an infinite dimensional function space of classical scalar fields $\phi\ :\ \mathcal{M}\ \to\ \mathbb{R}$. Heuristically, a preliminary although quite suggestive way to understand the functional measure $\mathfrak{D}\phi$ is in terms of

$$\int \mathfrak{D}\phi\ \ :=:\ \prod_{x\in\mathcal{M}}\int_{-\infty}^{\infty}\mathrm{d}\,\phi_x$$

Here :=: denotes a formal equality, the precise mathematical meaning of which has to be further specified, while the Minkowski spacetime point is treated as a discrete index, in such a way that the functional integral could be imagined as a continuous infinite generalization of a multiple Lebesgue integral. Taking the functional derivative operator $\mathrm{i}\ \delta/\delta J(x)$ through the functional Fourier integral (3.105) equation (3.101) becomes

$$\begin{aligned}0\ &=\ \left[(\,\Box_x + m^2\,)\,\frac{\mathrm{i}\,\delta}{\delta J(x)} + J(x)\right] Z_0[\,J\,]\\ &=\ \int \mathfrak{D}\phi\ \left[\frac{\delta\,S_0}{\delta\,\phi\,(x)}\ +\ J(x)\right]\widetilde{Z}_0[\,\phi\,]\ \exp\left\{\mathrm{i}\int \mathrm{d}y\ \phi(y)\,J(y)\right\}\end{aligned}$$

which enables us to identify

$$\widetilde{Z}_0[\,\phi\,] = \mathcal{N}\ \exp\left\{\mathrm{i}\,S_0[\,\phi\,]\right\} \tag{3.106}$$

$\mathcal{N}$ being any arbitrary classical external source independent quantity. As a matter of fact we have

$$\begin{aligned}&\int \mathfrak{D}\phi\ \left[\frac{\delta\,S_0}{\delta\,\phi\,(x)}\ +\ J(x)\right]\widetilde{Z}_0[\,\phi\,]\ \exp\left\{\mathrm{i}\int \mathrm{d}y\ \phi(y)\,J(y)\right\}\\ =\ &\int \mathfrak{D}\phi\ \left[\frac{\delta\,S_0}{\delta\,\phi\,(x)}\ +\ J(x)\right]\ \exp\left\{\mathrm{i}\,S_0[\,\phi\,] + \mathrm{i}\int \mathrm{d}y\ \phi(y)\,J(y)\right\}\\ =\ &\int \mathfrak{D}\phi\ \frac{-\,\mathrm{i}\,\delta}{\delta\,\phi\,(x)}\ \exp\left\{\mathrm{i}\,S_0[\,\phi\,] + \mathrm{i}\int \mathrm{d}y\ \phi(y)\,J(y)\right\}\end{aligned}$$

so that, if we assume the validity of the functional integration by parts, the very last expression formally yields

$$
\begin{aligned}
& \exp\left\{ -\mathrm{i}\left[\tfrac{1}{2}\,\phi_x \left(\Box_x + m^2 - \mathrm{i}\varepsilon \right) \phi_x - \phi_x\, J_x \right] \right\} \Big|_{\phi=-\infty}^{\phi=+\infty} \\
\times\; & \prod_{y\in\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\phi_y\; \exp\left\{ \mathrm{i}\, S_0[\,\phi\,] + \mathrm{i}\,\langle\, \phi_y\, J_y \,\rangle \right\} \Big]_{y\neq x} \;=\; 0
\end{aligned}
$$

the convergence factor being provided by the causal $+\mathrm{i}\varepsilon$ prescription, where we have taken into account that the boundary values for the codomain of the scalar field functional space are just

$$
-\infty < \phi_x < \infty \qquad\qquad \forall\, x \in \mathcal{M}
$$

while I have used the discrete index like notation

$$
\langle\, \phi_y\, J_y \,\rangle \;\equiv\; \int \mathrm{d}y\; \phi(y)\, J(y)
$$

A comparison with equation (3.102) leads to the formal equalities

$$
\begin{aligned}
Z_0[\,J\,] \;&=\; \exp\left\{ -\int \mathrm{d}x \int \mathrm{d}y\; \tfrac{1}{2}\, J(x)\, D_F(x-y)\, J(y) \right\} \\
&=:\; \mathcal{N} \int \mathfrak{D}\phi\; \exp\left\{ \mathrm{i}\, S_0[\,\phi\,] + \mathrm{i} \int \mathrm{d}x\; \phi(x)\, J(x) \right\} \\
&=:\; \mathcal{N} \prod_{x\in\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\phi_x\; \exp\left\{ -\tfrac{1}{2}\,\mathrm{i}\,\phi_x\, \mathcal{K}_x\, \phi_x + \mathrm{i}\,\phi_x\, J_x \right\}
\end{aligned}
\tag{3.107}
$$

where as usual

$$
S_0[\,\phi\,] = -\int \mathrm{d}x\; \tfrac{1}{2}\,\phi(x) \left(\Box + m^2 - \mathrm{i}\varepsilon \right) \phi(x) \;\equiv\; \langle\, -\tfrac{1}{2}\,\phi_x\, \mathcal{K}_x\, \phi_x \,\rangle
$$

The functional measure $\mathfrak{D}\phi$, which is a formal entity until now, can be implemented by the requirement of **invariance under field translations**

$$
\phi(x) \;\mapsto\; \phi'(x) \;=\; \phi(x) + f(x)
$$

Once it is assumed, after the change of variable

$$
\begin{aligned}
\phi(x) \;\mapsto\; \phi'(x) \;&=\; \phi(x) - (\,\Box + m^2 - \mathrm{i}\varepsilon\,)^{-1} J(x) \\
&=\; \phi(x) - \mathrm{i} \textstyle\int \mathrm{d}y\; D_F(x-y)\, J(y) \\
&\equiv\; \phi_x - \langle\, \mathrm{i}\, D_{xy}\, J_y \,\rangle
\end{aligned}
\tag{3.108}
$$

we find

$$
\begin{aligned}
Z_0[\,J\,] \;&:=:\; \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\; \tfrac{1}{2}\, J(x)\, D_F\,(x-y)\, J(y)\right\} \\
&\;\;\times\;\; \mathcal{N}\int \mathfrak{D}\phi\;\; \exp\left\{-\,\mathrm{i}\int \mathrm{d}x\; \tfrac{1}{2}\,\phi(x)\,(\,\Box + m^2 - \mathrm{i}\varepsilon\,)\;\phi(x)\right\} \\
&\;\;=\;\; Z_0[\,0\,]\, \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\; \tfrac{1}{2}\, J(x)\, D_F\,(x-y)\, J(y)\right\}
\end{aligned}
$$

`Proof` : from the change of the integration variable

$$
\phi_x \;\mapsto\; \phi'_x = \phi_x - \langle\, \mathrm{i}\, D_{xy}\, J_y\,\rangle \qquad\qquad \mathrm{d}\phi_x \;=\; \mathrm{d}\phi'_x
$$

equation (3.107) yields

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \mathrm{d}\phi_x\;\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\phi_x\,\mathcal{K}_x\,\phi_x + \mathrm{i}\,\phi_x\, J_x\right\} \\
=\;\; & \int_{-\infty}^{\infty} \mathrm{d}\phi'_x\;\; \exp\left\{-\,\mathrm{i}\,\tfrac{1}{2}\;(\phi'_x + \langle\,\mathrm{i}\,D_{xy}\,J_y\,\rangle)\;\mathcal{K}_x\;(\phi'_x + \langle\,\mathrm{i}\,D_{xz}\,J_z\,\rangle)\right\} \\
\times\;\; & \exp\left\{\,\mathrm{i}\,\phi'_x\,J_x -\; J_x\,\langle\, D_{xy}\,J_y\,\rangle\right\}
\end{aligned}
$$

Now, from the equality

$$
\mathcal{K}_x\,\langle\,\mathrm{i}\,D_{xz}\,J_z\,\rangle \;=\; J_x
$$

and the further equality

$$
\langle\,\mathrm{i}\,D_{xy}\,J_y\,\rangle\,\mathcal{K}_x\,\phi'_x \;\doteq\; \phi'_x\,\mathcal{K}_x\,\langle\,\mathrm{i}\,D_{xy}\,J_y\,\rangle \;=\; J_x\,\phi'_x
$$

which is true by negleting twice a boundary term, we can finally write

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \mathrm{d}\phi_x\;\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\phi_x\,\mathcal{K}_x\,\phi_x + \mathrm{i}\,\phi_x\,J_x\right\} \\
\doteq\;\; & \exp\left\{-\tfrac{1}{2}\,J_x\,\langle\, D_{xy}\,J_y\,\rangle\right\} \int_{-\infty}^{\infty} \mathrm{d}\phi'_x\;\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\phi'_x\,\mathcal{K}_x\,\phi'_x\right\} \qquad (\,\forall x \in \mathcal{M}\,)
\end{aligned}
$$

and thereby

$$
\begin{aligned}
& Z_0[\,J\,] :=: \mathcal{N}\prod_{x\in\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\phi_x\;\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\phi_x\,\mathcal{K}_x\,\phi_x + \mathrm{i}\,\phi_x\,J_x\right\} \\
\doteq\;\; & \prod_{x\in\mathcal{M}} \exp\left\{-\,\tfrac{1}{2}\,J_x\,\langle\, D_{xy}\,J_y\,\rangle\right\}\; \mathcal{N}\prod_{z\in\mathcal{M}} \int_{-\infty}^{\infty} \mathrm{d}\phi'_z\;\; \exp\left\{-\,\tfrac{1}{2}\,\mathrm{i}\,\phi'_z\,\mathcal{K}_z\,\phi'_z\right\} \\
=\;\; & \exp\left\{\langle\,-\,\tfrac{1}{2}\,J_x\,D_{xy}\,J_y\,\rangle\right\}\; \mathcal{N}\int \mathfrak{D}\phi'\;\; \exp\left\{\mathrm{i}S_0[\,\phi'\,]\right\} \\
=\;\; & \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\; \tfrac{1}{2}\,J(x)\,D_F\,(x-y)\,J(y)\right\}\;\; Z_0[\,0\,]
\end{aligned}
$$

Quod Erat Demonstrandum

Hence, self-consistency actually entails the formal identification

$$
Z_0[\,0\,] :=: \mathcal{N}\int \mathfrak{D}\phi\;\; \exp\left\{\mathrm{i}S_0[\,\phi\,]\right\} = 1
$$

As a matter of fact, from the very definition (3.95) it appears quite evident that $Z_0[\,0\,]$ is nothing but the vacuum to vacuum amplitude $\langle\,0\,|\,0\,\rangle$ that we suppose to be normalized to one. Thus, at this point, the strategy should be clear : if we were able to give a precise mathematical meaning to the above formal quantity $Z_0\,[\,0\,]$, then we will be able in turn to set up a mathematically sound and precise definition of the functional integral (3.105).

To this aim, let us first consider the euclidean formulation. Then we have to make the replacements

$$\mathrm{i}\,S_0[\,\phi\,]\ \longmapsto -S_E^{(0)}[\,\phi_E\,] \qquad x_{E\mu} = (\mathbf{x}\,,\,x_4 = \mathrm{i}\,x_0) \tag{3.109}$$

$$\begin{aligned} S_E^{(0)}[\,\phi_E\,] &= \int \mathrm{d}x_E\ \tfrac{1}{2}\Big(\partial_\mu\phi_E(x_E)\,\partial_\mu\phi_E(x_E) + m^2\,\phi_E^2(x_E)\Big) \\ &\doteq \int \mathrm{d}x_E\ \phi_E(x_E)\,\tfrac{1}{2}(m^2 - \partial_\mu\partial_\mu)\,\phi_E(x_E) \\ Z_E^{(0)}[\,0\,] &=: \mathcal{N}\int \mathfrak{D}\phi_E\ \exp\Big\{-S_E^{(0)}[\,\phi_E\,]\Big\} \end{aligned} \tag{3.110}$$

The above quantity is, formally, an absolutely convergent gaussian integral.

3.5.3 The Zeta Function Regularisation

According to the previously suggested heuristic interpretation, we could now understand the latter as

$$Z_E^{(0)}[\,0\,]\ =:\ \mathcal{N}\prod_x\int \mathrm{d}\phi_x\ \exp\left\{-\tfrac{1}{2}\,\phi_x\,K_E\,\phi_x\right\} \qquad K_E \equiv m^2 - \partial_\mu\partial_\mu$$

and if we assume that the functional integration variable can be rescaled

$$\phi_x\ \mapsto\ \phi_x' = \mu\,\phi_x$$

where μ is an arbitrary mass scale which does not influence the relevant $J(x)$ dependence of the generating functional, then we come to the expression

$$Z_E^{(0)}[\,0\,]\ =:\ \mathcal{N}'\prod_x\int_{-\infty}^{\infty} \mathrm{d}\phi_x'\ \exp\left\{-\tfrac{1}{2}\,\phi_x'\mu^{-2}\,K_E\,\phi_x'\right\} \tag{3.111}$$

in which the dimensionless, positive, second order and symmetric differential operator $\mu^{-2}\,(\,m^2 - \partial_\mu\partial_\mu\,)$ is involved, the spectrum of which is purely continuous and given by the positive eigenvalues $\mu^{-2}\,(\,m^2 + k_\mu k_\mu\,)$ with $k_\mu \in \mathbb{R}\ (\,\mu = 1,2,3,4\,)$. Then, for any finite dimensional real and symmetric $n\times n$ matrix $A = A^\top$, with $n\in\mathbb{N}$, it is well known that a real orthogonal matrix $R \in SO(n)$ always exists such that $R^\top A\,R = \mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ where

λ_i $(i = 1, 2, \ldots, n)$ are the real eigenvalues of the symmetric matrix. As a consequence we immediately obtain as a result of the gaussian integral

$$\begin{aligned} I &= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n \, \exp\{-\tfrac{1}{2}\, x^\top A x\} \\ &= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathrm{d}y_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}y_n \, \exp\{-\tfrac{1}{2}\, (Ry)^\top A\, Ry\} \\ &= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathrm{d}y_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}y_n \, \exp\left\{-\tfrac{1}{2} \sum_{i=1}^{n} \lambda_i\, y_i^2\right\} \\ &= (2\pi)^{-n/2} \prod_{i=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}y_i \, \exp\left\{-\tfrac{1}{2}\, \lambda_i\, y_i^2\right\} \\ &= (2\pi)^{-n/2} \prod_{i=1}^{n} \sqrt{\frac{2\pi}{\lambda_i}} \;=\; (\det A)^{-1/2} \end{aligned} \tag{3.112}$$

In view of this simple result we shall attempt to *define* the formal quantity $Z_E^{(0)}[\,0\,]$ to be precisely given by

$$\begin{aligned} Z_E^{(0)}[\,0\,] \;&=:\; \mathcal{N}' \prod_x \int_{-\infty}^{\infty} \mathrm{d}\phi'_x \, \exp\left\{-\tfrac{1}{2}\, \phi'_x\, \mu^{-2}\, K_E\, \phi'_x\right\} \\ &\equiv\; \mathcal{N}' \det \|\, \mu^{-2}\, (m^2 - \partial_\mu \partial_\mu)\, \|^{-1/2} \end{aligned} \tag{3.113}$$

where the determinant of a positive symmetric differential operator can be suitably defined by means of the so called Zeta function regularization :

Steven W. Hawking
Zeta function regularization of path integrals in curved space time
Commun. Math. Phys. **55** (1977) 133

The idea beyond this method is as simple as powerful and is based upon the analytic continuation tool. Consider as an example a positive operator $A > 0$ with a purely discrete spectrum, so that its spectral decomposition reads

$$A = \sum_{k=1}^{\infty} \lambda_k\, P_k \qquad \lambda_k > 0 \quad \mathrm{tr}\, P_k = d_k < \infty \quad \forall\, k \in \mathbb{N}$$

The complex powers of the positive operator $A > 0$ can be easily obtained in terms of its spectral resolution

$$A^{-s} = \sum_{k=1}^{\infty} \lambda_k^{-s}\, P_k = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\; t^{\,s-1} \exp\{-t\lambda_k\}\, P_k \qquad \Re\mathrm{e}\, s > 0$$

Let us now further suppose the positive operator A to be compact and of the trace class. Hence, from the spectral decomposition theorem we can write the integral kernel, or Green function,

$$\langle x\,|\,A^{-s}\,|\,y\rangle = \sum_{k=1}^{\infty}\lambda_k^{-s}\,\langle x\,|\,P_k\,|\,y\rangle = \sum_{k=1}^{\infty}\lambda_k^{-s}\,\psi_k(x)\,\psi_k^*(y) \tag{3.114}$$

$$A^{-s}\,\psi_k(x) = \lambda_k^{-s}\,\psi_k(x) \qquad \int \mathrm{d}x\;\psi_k(x)\,\psi_n^*(x) = \delta_{kn} \tag{3.115}$$

$$\begin{aligned}
\mathrm{Tr}\,A^{-s} &= \int \mathrm{d}x\;\langle x\,|\,A^{-s}\,|\,x\rangle = \sum_{k=1}^{\infty}\lambda_k^{-s}\,d_k \\
&= \sum_{k=1}^{\infty} d_k\,\frac{1}{\Gamma(s)}\int_0^{\infty}\mathrm{d}t\;t^{\,s-1}\,\exp\{-t\lambda_k\} \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty}\mathrm{d}t\;t^{\,s-1}\sum_{k=1}^{\infty} d_k\,\exp\{-t\lambda_k\} < \infty
\end{aligned} \tag{3.116}$$

always in the strip $\Re\mathrm{e}\,s > 0$ of the complex plane. Now we have

$$\frac{d}{ds}\,\mathrm{Tr}\,A^{-s} = \sum_{k=1}^{\infty}\frac{d}{ds}\,\lambda_k^{-s}\,d_k = \sum_{k=1}^{\infty}(-\,\ln\lambda_k)\,\lambda_k^{-s}\,d_k \tag{3.117}$$

and thereby we obtain the zeta function regularization of the determinant of a positive and compact operator of the trace class : namely,

$$\ln\det A = \sum_{k=1}^{\infty} d_k\,\ln\lambda_k \;\stackrel{\mathrm{def}}{=}\; -\,\frac{d}{ds}\,\mathrm{Tr}\,A^{-s}\,\Big]_{s=0} \tag{3.118}$$

provided the analytic continuation is possible to include the imaginary axis $\Re\mathrm{e}\,s = 0$, *via* some suitable deformation of the integration path in complex plane, just like in the original case of the Riemann's Zeta function [4]

$$\zeta(s,q) = -\,\frac{\Gamma(1-s)}{2\pi\mathrm{i}}\int_{\infty}^{(0+)}\mathrm{d}\theta\;\frac{(-\theta)^{\,s-1}\,e^{\,-q\theta}}{1-e^{\,-\theta}} \tag{3.119}$$

in which it is assumed that the path of integration does not pass through the points $2n\pi\mathrm{i}$ where n is a natural number.

[4]Gradshteyn and Ryzhik [22] § **9.5** pp.1100-1103 ; Bruno Pini (1979) *Lezioni sulle distribuzioni, 1. Distribuzioni temperate*, § **4** Appendice, cap. **3.** p. 276.

In order to apply the above treatment to the case of interest, one is faced with the problem that the euclidean Klein-Gordon operator is neither compact nor of the trace class. To overcome this difficulty, it is expedient to introduce a very large box, *e.g.* a symmetric hypercube of side $2L$, to impose periodic boundary conditions on its faces and to make eventually the transition to the infinite volume continuum limit. In the presence of a symmetric hypercube with periodic boundary conditions, the spectrum of the euclidean Klein-Gordon operator is purely discrete and nondegenerate

$$\lambda_n = m^2 + \frac{\pi^2}{L^2}\, n_\mu n_\mu \qquad n_\mu \in \mathbb{Z} \quad \mu = 1,2,3,4 \tag{3.120}$$

and in the limit of $L \to \infty$ we can safely replace

$$\sum_{n_\mu = -\infty}^{\infty} \quad \longmapsto \quad 2L \int_{-\infty}^{\infty} \frac{\mathrm{d}\,k_\mu}{2\pi} \qquad (\mu = 1,2,3,4)$$

$$\sum_n \quad \longmapsto \quad V \int \frac{\mathrm{d}^4 k}{(2\pi)^4}$$

Then we find for $A = \mu^{-2}(m^2 - \partial_\mu \partial_\mu)$

$$\begin{aligned}
\mathrm{Tr}\, A^{-s} &\doteq V\,\mu^{2s} \int \frac{\mathrm{d}\,k}{(2\pi)^4}\,(m^2 + k^2)^{-s} \\
&= \frac{V\,\mu^{2s}}{16\pi^2\,\Gamma(s)} \int_0^\infty \mathrm{d}\,t\; t^{s-3}\, \exp\{-t m^2\} \\
&= \frac{V m^4}{16\pi^2} \left(\frac{\mu}{m}\right)^{2s} \frac{\Gamma(s-2)}{\Gamma(s)} \\
&= \frac{V m^4}{16\pi^2} \left(\frac{\mu}{m}\right)^{2s} (s^2 - 3s + 2)^{-1}
\end{aligned} \tag{3.121}$$

where $\doteq$ means that the transition to the continuum limit is understood. Hence

$$\begin{aligned}
\frac{d}{ds}\,\mathrm{Tr}\, A^{-s} &= (s^2 - 3s + 2)^{-1}\,\mathrm{Tr}\, A^{-s} \\
&\times \left[2(s^2 - 3s + 2)\, \ln\frac{\mu}{m} - 2s + 3 \right]
\end{aligned} \tag{3.122}$$

and thereby

$$\det \| \left(m^2 - \partial_\mu \partial_\mu\right)/\mu^2 \| = \exp\left\{ \frac{V m^4}{16\pi^2} \left(\ln\frac{m}{\mu} - \frac{3}{4} \right) \right\} \tag{3.123}$$

Turning back to equation (3.113) we see that

$$Z_E^{(0)}[\,0\,] = 1 \qquad \Longleftrightarrow \qquad \mathcal{N}' \equiv \exp\left\{\frac{Vm^4}{32\pi^2}\left(\ln\frac{m}{\mu} - \frac{3}{4}\right)\right\}$$

and the transition to the Minkowski spacetime can be immediately done by simply replacing $\quad V_{\mathrm{euclidean}} \quad \leftrightarrow \quad \mathrm{i}\,V_{\mathrm{minkowskian}}$

The conclusion of all the above formal reasoning is as follows : we are enabled to *define the functional integral* for a free scalar field theory by the equalities

$$\begin{aligned}
Z_0[\,J\,] &= \exp\left\{-\frac{1}{2}\int \mathrm{d}x \int \mathrm{d}y\ J(x)\,D_F\,(x-y)\,J(y)\right\} \\
&\stackrel{\mathrm{def}}{=} \mathcal{N}\int \mathfrak{D}\phi\ \exp\left\{\mathrm{i}\,S_0\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\ \phi(x)\,J(x)\right\} \\
S_0\,[\,\phi\,] &= -\int \mathrm{d}x\,\tfrac{1}{2}\,\phi(x)\,\big(\Box + m^2 - \mathrm{i}\varepsilon\big)\ \phi(x) \\
\mathcal{N} &= \mathrm{constant} \times \Big(\det\,\|\,\Box + m^2\,\|\Big)^{1/2} \\
&\stackrel{\mathrm{def}}{=} \exp\left\{\frac{\mathrm{i}\,Vm^4}{32\pi^2}\left(\ln\frac{m}{\mu} - \frac{3}{4}\right)\right\} \qquad \text{(Zeta regularization)} \\
Z_0\,[\,0\,] &= \mathcal{N}\int \mathfrak{D}\phi\ \exp\{\mathrm{i}\,S_0\,[\,\phi\,]\} = 1
\end{aligned}$$

Notice that the integral kernel $D_F\,(x-y)$ which appears in the exponent of the right hand side of eq. (3.102) is just the opposite of the inverse for the kinetic operator $-\,\mathrm{i}\,(\,\Box + m^2\,)$ that specifies the exponent $\mathrm{i}\,S_0\,[\,\phi\,]$ of the functional integrand.

To summarize the above long discussion concerning the meaning and the construction of the functional integration, I can list a number of key features. The functional integral does fulfil by construction the following properties :

1. linearity

$$\int \mathfrak{D}\phi\ (\alpha\,F\,[\,\phi\,] + \beta\,G\,[\,\phi\,]) = \alpha\int \mathfrak{D}\phi\ F\,[\,\phi\,] + \beta\int \mathfrak{D}\phi\ G\,[\,\phi\,]$$

   for any pair of complex functions $\alpha,\beta\,:\,\mathcal{M}\,\to\,\mathbb{C}$

2. translation invariance

$$\int \mathfrak{D}\phi\ F\,[\,\phi + \alpha\,] = \int \mathfrak{D}\phi\ F\,[\,\phi\,] \qquad \forall\,\alpha\,:\,\mathcal{M}\,\to\,\mathbb{C}$$

3. rescaling

$$\int \mathfrak{D}\phi\ F\,[\,(\,A\phi\,)\,(x)\,] = (\det A)^{-1} \int \mathfrak{D}\phi\ F\,[\,\phi\,]$$

where A is any invertible integro-differential operator

4. integration by parts

$$0 = \int \mathfrak{D}\phi\ \frac{\delta F\,[\,\phi\,]}{\delta\,\phi(x)}\,G\,[\,\phi\,] + \int \mathfrak{D}\phi\ F\,[\,\phi\,]\,\frac{\delta G\,[\,\phi\,]}{\delta\,\phi(x)}$$

The above properties 1.– 4. are valid for any functional integrand F, G of the gaussian kind $\mathcal{P}\,[\,\phi\,]\ \exp\left\{\mathrm{i}\,S_0\,[\,\phi\,] + \mathrm{i}\int \mathrm{d}x\ \phi(x)\,J(x)\right\}$ $(n \in \mathbb{N})$ with $\mathcal{P}\,[\,\phi\,]$ any polynomial functional of the scalar field and its derivatives.

References

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3.6 Problems

1. **The complex scalar field.** Consider the field theory of a complex valued free scalar field with the Lagrange density

$$\mathcal{L}(x) = \partial_\mu \phi^*(x)\, \partial^\mu \phi(x) - m^2\, \phi^*(x)\, \phi(x)$$

It is easier to analyse the theory by considering $\phi(x)$ and $\phi^*(x)$ as the independent variables in configuration space rather than the real and imaginary parts of the complex scalar field function.

(**a**) Find the hamiltonian and the canonical equations of motion

Solution. The action is given by

$$S[\,\Phi\,] = \int \mathrm{d}x\ \mathcal{L}[\,\Phi(x)\,] \qquad \Phi(x) = u(x) + i\,v(x)$$

which leads to the conjugated canonical momenta

$$\frac{\delta S}{\delta\,\dot{\Phi}(x)} = \partial_0 \Phi^*(x) \equiv \Pi(x) \qquad \frac{\delta S}{\delta\,\dot{\Phi}^*(x)} = \partial_0 \Phi(x) \equiv \Pi^*(x)$$

and to the classical hamiltonian functional

$$\begin{aligned} H[\,\Pi, \Phi\,] &= \int \mathrm{d}\mathbf{x}\, \Big(\Pi(x)\,\dot{\Phi}(x) + \Pi^*(x)\,\dot{\Phi}^*(x) - \mathcal{L}(x)\Big) \\ &= \int \mathrm{d}\mathbf{x}\, \Big(\,|\,\Pi(x)\,|^2 + |\,\boldsymbol{\nabla}\Phi(x)\,|^2 + m^2\,|\,\Phi(x)\,|^2\Big) \end{aligned}$$

The Poisson brackets are evidently given by

$$\{\Phi(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \delta\,(\mathbf{x}-\mathbf{y}) = \{\Phi^*(t,\mathbf{x})\,,\,\Pi^*(t,\mathbf{y})\}$$

all the others being equal to zero, so that the Hamilton equations read

$$\begin{cases} \dot{\Phi}(x) = \{\Phi(x)\,,\,H\} = \Pi(x) \\ \dot{\Pi}(x) = \{H\,,\,\Pi(x)\} = \ddot{\Phi}(x) \end{cases}$$

$$\begin{cases} \dot{\Phi}^*(x) = \{\Phi(x)^*, H\} = \Pi^*(x) \\ \dot{\Pi}^*(x) = \{H, \Pi^*(x)\} = \ddot{\Phi}^*(x) \end{cases}$$

whence we immediately find the Klein-Gordon wave equations

$$\ddot{\Phi}(x) = \boldsymbol{\nabla}^2\Phi(x) - m^2\,\Phi(x) \qquad (\,\Box + m^2\,)\,\Phi(x) = 0$$

(**b**) Diagonalize the hamiltonian operator introducing creation and annihilation operators. Show that the complex scalar free field contains two types of massive spinless particles of rest mass m.

Solution. The normal mode decomposition of the complex scalar free field can be easily obtained by a straightforward generalization of the treatment for the real scalar free field (3.42). The result is evidently

$$
\begin{aligned}
\Phi(x) &= \sum_{\mathbf{k}} \Big[a_{\mathbf{k}}\, u_{\mathbf{k}}(x) + b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \Big] \\
\Pi(x) &= \sum_{\mathbf{k}} i\omega_{\mathbf{k}} \Big[-\, b_{\mathbf{k}}\, u_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \Big] \\
u_{\mathbf{k}}(x) &\equiv [\, 2\omega_{\mathbf{k}}(2\pi)^3 \,]^{-1/2}\ \exp\{-\, i\, x^0 \omega_{\mathbf{k}} + i\, \mathbf{k}\cdot\mathbf{x}\}
\end{aligned}
$$

where $\omega_{\mathbf{k}} = (\, \mathbf{k}^2 + m^2 \,)^{1/2}$ together with the canonical commutation relations

$$
\begin{aligned}
[\, \Phi(t,\mathbf{x})\,,\, \Phi(t,\mathbf{y}) \,] = 0 &= [\, \Pi(t,\mathbf{x})\,,\, \Pi(t,\mathbf{y}) \,] \\
[\, \Phi(t,\mathbf{x})\,,\, \Pi(t,\mathbf{y}) \,] \ &= \ i\,\delta\,(\mathbf{x}-\mathbf{y})
\end{aligned}
$$

It is clear that the main difference with respect to the real case is the appearence of two kinds of creation and destruction operators, as the reality conditions no longer hold true, which satisfy the algebra

$$
\begin{aligned}
&[\, a_{\mathbf{k}}\,, a_{\mathbf{p}} \,] = [\, b_{\mathbf{k}}\,, b_{\mathbf{p}} \,] = 0 \\
&[\, a_{\mathbf{k}}\,, b_{\mathbf{p}} \,] = [\, a_{\mathbf{k}}^{\dagger}\,, b_{\mathbf{p}} \,] = 0 \\
&[\, a_{\mathbf{k}}\,, a_{\mathbf{p}}^{\dagger} \,] = [\, b_{\mathbf{k}}\,, b_{\mathbf{p}}^{\dagger} \,] = \delta\,(\mathbf{k}-\mathbf{p})
\end{aligned}
$$

Then the normal ordered hamiltonian and momentum operator takes the diagonal form

$$
\begin{aligned}
H\,[\,\Pi,\Phi\,] \ &= \ \int \mathrm{d}\mathbf{x}\ :|\,\Pi(x)\,|^2 + |\,\boldsymbol{\nabla}\Phi(x)\,|^2 + m^2\,|\,\Phi(x)\,|^2 : \\
&= \ \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) = P_0 \\
\mathbf{P} \ &= \ -\int \mathrm{d}\mathbf{x}\ \ :\Pi(x)\boldsymbol{\nabla}\Phi(x) + \Pi^{*}(x)\boldsymbol{\nabla}\Phi^{*}(x) : \\
&= \ \sum_{\mathbf{k}} \mathbf{k} \left(a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right)
\end{aligned}
$$

It follows therefrom that the complex scalar free field describe two kinds of particles with the very same value m of the rest mass.

(**c**) Rewrite the conserved Noether charge

$$Q = \mathrm{i}\int \mathrm{d}\mathbf{x}\ \Big[\,\Phi^*(x)\Pi^*(x) - \Phi(x)\Pi(x)\,\Big]$$

in terms of creation and annihilation operators and evaluate the charge of the particles of each type.

Solution. From the invariance of the classical lagrangian under $U(1)$ phase transformations $\Phi(x) \mapsto \Phi'(x) = e^{-i\alpha}\Phi(x)$ we immediately get the above written Noether charge. Moreover, from the normal mode expansion and the normal ordering prescription we readily obtain

$$\begin{aligned} Q &= \mathrm{i}\int \mathrm{d}\mathbf{x}\ :\Phi^*(x)\Pi^*(x) - \Phi(x)\Pi(x): \\ &= \sum_{\mathbf{k}} \Big(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} - b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \Big) \end{aligned}$$

which is understood so that each particle normal mode carries one unit of positive charge, whereas each antiparticle normal mode carries one unit of negative charge, the sign of the charge being conventional.

2. **Poincaré covariance**

(**a**) The four energy momentum operators P_μ ($\mu = 0,1,2,3$) and the six orbital angular momentum operators $L_{\mu\nu} = -L_{\nu\mu}$ are the infinitesimal operators, or generators, of the Poincaré group, for the infinite dimensional unitary representation acting on the Fock space of a real scalar quantized field $\phi(x)$.

(**a**) Show that $[\,\phi(x)\,,\,L_{\mu\nu}\,] = \mathrm{i}\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu)\,\phi(x)$

Solution. We have

$$T^{\,0}_{\ \mu}(x) = \Pi\,(x)\,\partial_\mu\,\phi\,(x) - \delta^0_\mu\,\mathcal{L}\,(x)$$

and the equal–time canonical commutation relations

$$[\,\phi\,(t,\mathbf{x})\,,\,\mathcal{L}\,(t,\mathbf{y})\,] = \Pi\,(t,\mathbf{y})\,[\,\phi\,(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] = \mathrm{i}\,\Pi\,(t,\mathbf{x})\,\delta\,(\mathbf{x}-\mathbf{y})$$

$$[\,\phi\,(t,\mathbf{x})\,,\Pi\,(t,\mathbf{y})\,\partial_\nu\,\phi\,(t,\mathbf{y})\,] = \mathrm{i}\,\Big(\partial_\nu\,\phi\,(t,\mathbf{y}) + \delta^0_\nu\,\Pi\,(t,\mathbf{y})\Big)\,\delta\,(\mathbf{x}-\mathbf{y})$$

Then we obtain for $x_0 = y_0 = t$

$$\begin{aligned} [\,\phi\,(x)\,,\,L_{\mu\nu}\,] &= \int \mathrm{d}\mathbf{y}\ y_\mu\,[\,\phi\,(t,\mathbf{x})\,,\,T^{\,0}_{\ \nu}\,(t,\mathbf{y})\,] - \mu \leftrightarrow \nu \\ &= \mathrm{i}\,x_\mu\,\partial_\nu\,\phi\,(t,\mathbf{x}) - \mathrm{i}\,x_\nu\,\partial_\mu\,\phi\,(t,\mathbf{x}) \end{aligned}$$

and thereby

$$\frac{\mathrm{i}}{2}\,[\,\phi\,(x)\,,\,L_{\mu\nu}\,]\,\delta\,\omega^{\,\mu\nu} = x_\nu\,\partial_\mu\,\phi\,(x)\,\epsilon^{\,\mu\nu} = \delta\,x^{\mu}\,\partial_\mu\,\phi\,(x)$$

It follows that the finite passive Lorentz transformations for the real quantized scalar field read

$$\phi'(x) = U(\omega)\,\phi\,(x)\,U^{\,\dagger}(\omega) = \phi\,(\Lambda\,x)$$

where

$$U(\omega) = \exp\left\{-\,\frac{\mathrm{i}}{2}\,\omega^{\,\mu\nu}\,L_{\mu\nu}\right\}$$

(**b**) Show that $[\,L_{\mu\nu}\,,\,P_\rho\,] = -\,\mathrm{i}\,g_{\,\mu\rho}\,P_\nu + \mathrm{i}\,g_{\,\nu\rho}\,P_\mu\,.$

Solution.

Let us first calculate the commutator $[\,L_{\mu\nu}\,,\,P_0\,]$. To this aim, for any analytic functional of the scalar field $\phi\,(x)$ and conjugated momentum $\Pi\,(x)$ operators we have

$$[\,\mathfrak{F}\,(\phi\,(x), \Pi\,(x))\;,\;P_\mu\,] = \mathrm{i}\,\partial_\mu\,\mathfrak{F}$$

In fact, for example,

$$[\,\phi^2\,(x)\,,\,P_\mu\,] = 2\phi\,(x)\,[\,\phi\,(x)\,,\,P_\mu\,] = 2\phi\,(x)\,\mathrm{i}\partial_\mu\,\phi\,(x) = \mathrm{i}\partial_\mu\,\phi^2\,(x)$$

and iterating we obviously get $\forall\,n \in \mathbb{N}$

$$[\,\phi^{\,n}\,(x)\,,\,P_\mu\,] = \mathrm{i}\,\partial_\mu\,\phi^{\,n}\,(x)$$

$$[\,\Pi^{\,n}\,(x)\,,\,P_\mu\,] = \mathrm{i}\,\partial_\mu\,\Pi^{\,n}\,(x)$$

so that the above statement holds true. Then we find

$$\begin{aligned}
[\,L_{\mu\nu}\,,\,P_0\,] &= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,[\,T^{\,0}_{\ \nu}(x)\,,\,P_0\,] - x_\nu\,[\,T^{\,0}_{\ \mu}(x)\,,\,P_0\,]\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,\mathrm{i}\partial_0\,T^{\,0}_{\ \nu}(x) - x_\nu\,\mathrm{i}\partial_0\,T^{\,0}_{\ \mu}(x)\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(-\;x_\mu\,\mathrm{i}\partial_\jmath\,T^{\,\jmath}_{\ \nu}(x) + x_\nu\,\mathrm{i}\partial_\jmath\,T^{\,\jmath}_{\ \mu}(x)\Big) \\
&\doteq \mathrm{i}\int \mathrm{d}\mathbf{x}\,\Big(g_{\,\mu\jmath}\,T^{\,\jmath}_{\ \nu}(x) - g_{\,\nu\jmath}\,T^{\,\jmath}_{\ \mu}(x)\Big) \\
&= \mathrm{i}\int \mathrm{d}\mathbf{x}\,\Big(T_{\mu\nu}(x) - g_{\,\mu 0}\,T^{\,0}_{\ \nu}(x) - T_{\nu\mu}(x) - g_{\,\nu 0}\,T^{\,0}_{\ \mu}(x)\Big) \\
&= -\,\mathrm{i}\,g_{\,\mu 0}\,P_\nu + \mathrm{i}\,g_{\,\nu 0}\,P_\mu
\end{aligned}$$

in which, as usual, a boundary term has been neglected. Furthermore we find

$$\begin{aligned}
[\,L_{\mu\nu}\,,\,P_\jmath\,] &= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,[\,T^{\,0}_{\;\nu}(x)\,,\,P_\jmath\,] - x_\nu\,[\,T^{\,0}_{\;\mu}(x)\,,\,P_\jmath\,]\Big) \\
&= \int \mathrm{d}\mathbf{x}\,\Big(x_\mu\,\mathrm{i}\partial_\jmath\,T^{\,0}_{\;\nu}(x) - x_\nu\,\mathrm{i}\partial_\jmath\,T^{\,0}_{\;\mu}(x)\Big) \\
&\doteq -\,\mathrm{i}\int \mathrm{d}\mathbf{x}\,\Big(g_{\mu\jmath}\,T^{\,0}_{\;\nu}(x) - g_{\nu\jmath}\,T^{\,0}_{\;\mu}(x)\Big) \\
&= -\,\mathrm{i}\,g_{\mu\jmath}\,P_\nu + \mathrm{i}\,g_{\nu\jmath}\,P_\mu
\end{aligned}$$

up to a boundary term, which completes the proof.

(**c**) A fully detailed check of the canonical commutation relations

$$[\,L_{\mu\nu}\,,\,L_{\rho\sigma}\,] = -\,\mathrm{i}\,g_{\mu\rho}\,L_{\nu\sigma} + \mathrm{i}\,g_{\nu\rho}\,L_{\mu\sigma} + \mathrm{i}\,g_{\mu\sigma}\,L_{\nu\rho} - \mathrm{i}\,g_{\nu\sigma}\,L_{\mu\rho}$$

is straightforward although somewhat tedious and can be found in [10], **7.8**, pp. 144 - 147.

3. **Special distributions in configuration space.**

(**a**) Evaluate the scalar distribution

$$i\,\langle\,0\,|\,\phi(x)\,\phi(y)\,|\,0\,\rangle = i\,[\,\phi^{(-)}(x)\,,\phi^{(+)}(y)\,] \equiv D^{(-)}(x-y)$$

explicitly in terms of Bessel functions.

Solution. Let us consider the positive and negative parts of the Pauli-Jordan distribution (3.82) in the four dimensional Minkowski space-time: namely,

$$\begin{aligned}
D^{(\pm)}(x) &\equiv \frac{\pm 1}{(2\pi)^3\,i}\int \exp\{\pm\,i\,k\cdot x)\}\,\delta\,(k^2-m^2)\,\theta(k_0)\,d^4k \\
&= \frac{\pm 1}{(2\pi)^3\,2i}\int \mathrm{d}\mathbf{k}\,\left(\mathbf{k}^2+m^2\right)^{-1/2} \\
&\times\ \exp\{\pm\,i\,x^0(\mathbf{k}^2+m^2)^{1/2} \mp i\,\mathbf{k}\cdot\mathbf{x}\} \\
&\equiv \frac{\pm 1}{4i\pi^2\,r}\int_0^\infty \mathrm{d}k\,\frac{k\,\sin(kr)}{(k^2+m^2)^{1/2}} \\
&\times\ \exp\left\{\pm\,i\,x^0\,(k^2+m^2)^{1/2}\,\right\} \\
&= \frac{\pm i}{4\pi^2\,r}\,\frac{d}{dr}\int_0^\infty \mathrm{d}k\,\frac{\cos(kr)}{(k^2+m^2)^{1/2}} \\
&\times\ \exp\left\{\pm\,i\,x^0\,(k^2+m^2)^{1/2}\,\right\}
\end{aligned}$$

where $r \equiv |\,\mathbf{x}\,|,\ k = |\,\mathbf{k}\,|$, whence it is clear that the positive and negative parts of the Pauli-Jordan commutator are complex conjugate quantities

$$[\,D^{(\pm)}(x)\,]^* = D^{(\mp)}(x)$$

Then we can write

$$\begin{aligned} D^{(+)}(x) &= \frac{\mathrm{i}}{8\pi^2\, r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{\cos(kr)}{(k^2+m^2)^{1/2}} \\ &\times\ \exp\left\{i\,x^0\,(k^2+m^2)^{1/2}\right\} \\ &= \frac{i}{8\pi^2\, r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\int_{-\infty}^{\infty}\mathrm{d}k\ (k^2+m^2)^{-1/2} \\ &\times\ \exp\left\{i\,x^0\,(k^2+m^2)^{1/2}+i\,kr\right\} \end{aligned}$$

Consider now the integral

$$\begin{aligned} I(x^0,r) &= \int_{-\infty}^{\infty}\mathrm{d}k\ (k^2+m^2)^{-1/2} \\ &\times\ \exp\{i\,x^0\,(k^2+m^2)^{1/2}+i\,kr\} \qquad (x^0>0) \end{aligned}$$

and perform the change of variable $k = m\sinh\eta$, so that $(k^2+m^2)^{1/2} = m\cosh\eta$. Then we obtain

$$I(x^0,r) = \int_{-\infty}^{\infty} d\eta\ \exp\{im\,(x^0\cosh\eta + r\sinh\eta)\}$$

Here $x^0 > 0$ so that two cases should be distinguished, *i.e.* $0 < x^0 < r$ and $x^0 > r$. By setting $\lambda \equiv (x^0)^2 - \mathbf{x}^2$ it is convenient to carry out respectively the substitutions

$$\begin{cases} x^0 = \sqrt{-\lambda}\,\sinh\xi\,, & r = \sqrt{-\lambda}\,\cosh\xi\,,\quad 0 < x^0 < r \\ x^0 = \sqrt{\lambda}\,\cosh\xi\,, & r = \sqrt{\lambda}\,\sinh\xi\,, \qquad x^0 > r \end{cases}$$

in such a way that we can write

$$\begin{aligned} I(x^0,r) &= \theta(-\lambda)\int_{-\infty}^{\infty} d\eta\ \exp\{im\sqrt{-\lambda}\,\sinh(\xi+\eta)\} \\ &+\ \theta(\lambda)\int_{-\infty}^{\infty} d\eta\ \exp\{im\sqrt{\lambda}\,\cosh(\xi+\eta)\} \\ &= \theta(-\lambda)\int_{-\infty}^{\infty} d\eta\ \exp\{im\sqrt{-\lambda}\,\sinh\eta\} \\ &+\ \theta(\lambda)\int_{-\infty}^{\infty} d\eta\ \exp\{im\sqrt{\lambda}\,\cosh\eta\} \qquad (x^0>0) \end{aligned}$$

Now we can use the integral representations of the cylindrical Bessel functions of real and imaginary arguments [22] eq.s **8.421**1. p. 965 and **8.432**4. p. 969 that yield

$$\begin{aligned}
I(x^0, r) &= \theta(x^0)\left[2\theta(-\lambda)\, K_0(m\sqrt{-\lambda}) + \theta(\lambda)\,\pi i\, H_0^{(1)}(m\sqrt{\lambda})\right] \\
&+ \theta(-x^0)\left[2\theta(-\lambda)\, K_0(m\sqrt{-\lambda}) - \theta(\lambda)\,\pi\, i\, H_0^{(2)}(m\sqrt{\lambda})\right] \\
&= 2\theta(-\lambda)\, K_0(m\sqrt{-\lambda}) \\
&+ \pi\,\theta(\lambda)\left[i\,\mathrm{sgn}(x^0)\, J_0(m\sqrt{\lambda}) - N_0(m\sqrt{\lambda})\right]
\end{aligned}$$

and finally

$$D^{(+)}(x) = \frac{i}{8\pi^2 r}\,\frac{d}{dr}\, I(x^0, r) = \frac{1}{4\, i\, \pi^2}\,\frac{d}{d\lambda}\, I(x^0, \lambda)$$

We note that in the neighborhood of the origin the cylindrical Bessel functions of real and imaginary arguments may be represented in the form

$$\begin{aligned}
J_0(z) &= 1 - \left(\frac{z}{2}\right)^2 + O(z^4) \\
N_0(z) &= \frac{2}{\pi}\left[1 - \left(\frac{z}{2}\right)^2\right] \ln\frac{z}{2} + \frac{2}{\pi}\,\mathbf{C} + O(z^2) \\
K_0(z) &= -\left[1 + \left(\frac{z}{2}\right)^2\right] \ln\frac{z}{2} - \mathbf{C} + O(z^2)
\end{aligned}$$

where $\mathbf{C}$ is the Mascheroni's constant. By replacing the differentiation with respect to x with differentiation with respect to λ and taking into account the discontinuity of the function $I(x^0, x)$ on the light-cone manifold $\lambda = 0$ we obtain the following expression for the positive part of the Pauli-Jordan commutator

$$\begin{aligned}
D^{(+)}(x) &= \frac{1}{4\pi}\,\mathrm{sgn}(x^0)\,\delta(\lambda) - i\,\theta(-\lambda)\,\frac{m}{4\pi^2\sqrt{-\lambda}}\, K_1(m\sqrt{-\lambda}) \\
&- i\,\theta(\lambda)\,\frac{m}{8\pi\sqrt{\lambda}}\left[N_1(m\sqrt{\lambda}) - i\,\mathrm{sgn}(x^0)\, J_1(m\sqrt{\lambda})\right]
\end{aligned}$$

so that the Pauli-Jordan commutator and the Feynman propagator are respectively expressed by

$$\begin{aligned}
D(x) &= D^{(+)}(x) + D^{(-)}(x) \\
&= \frac{1}{2\pi}\,\mathrm{sgn}(x^0)\,\delta(\lambda) - \frac{m}{4\pi\sqrt{\lambda}}\,\theta(\lambda)\,\mathrm{sgn}(x^0)\, J_1(m\sqrt{\lambda})
\end{aligned}$$

$$
\begin{aligned}
D_F(x) &= i\,\theta(-x^0)\,D^{(+)}(x) - i\,\theta(x^0)\,D^{(-)}(x) \\
&= \frac{1}{4\pi i}\;\delta(\lambda) + \frac{m\,\theta(-\lambda)}{4\pi^2\sqrt{-\lambda}}\;K_1(m\sqrt{-\lambda}) \\
&+\;\theta(\lambda)\,\frac{m}{8\pi\sqrt{\lambda}}\;\Big[\,N_1(m\sqrt{\lambda}) + i\,J_1(m\sqrt{\lambda})\,\Big]
\end{aligned}
$$

(**b**) Evaluate the scalar causal 2-point Green function of order n in the $D-$dimensional Minkowski spacetime, which is defined to be

$$
G_n^{(D)}(z) = \frac{i}{(2\pi)^D}\int\frac{\exp\{-\,i\,k\cdot z\}}{(k^2-m^2+i\varepsilon)^n}\;d^Dk
$$

explicitly in terms of Bessel functions.

Solution. It is very instructive to first compute the integral

$$
\begin{aligned}
I_n^D(z) &\equiv \frac{i}{(2\pi)^D}\int\frac{\exp\{-\,i\,k\cdot z\}}{(k^2-m^2+i\varepsilon)^n}\;d^Dk \\
&= \frac{i\,(2m)^{1-n}}{(2\pi)^D\,(n-1)!}\left(\frac{d^{n-1}}{dm^{n-1}}\int\frac{\exp\{-\,i\,k\cdot z\}}{k^2-m^2+i\varepsilon}\;d^Dk\right)
\end{aligned}
$$

where $z=(z^0,z^1,\ldots,z^{D-1})$ and $k=(k^0,k^1,\ldots,k^{D-1})$ are coordinate and conjugate momentum in a D-dimensional Minkowski spacetime, so that $k\cdot z = k^0z^0 - k^1z^1 - \cdots - k^{D-1}z^{D-1} = k^0z^0 - \mathbf{k}\cdot\mathbf{z}$, while n is a sufficiently large natural number that will be better specified further on. Turning to a D-dimensional euclidean space, after setting $z^0 = iz_D\,,\ k^0 = ik_D$ we immediately obtain

$$
I_n^D(z) \equiv \frac{(-1)^n}{(2\pi)^D}\int\frac{\exp\{i\,k_E\cdot z_E\}}{(k_E^2+m^2)^n}\;d^Dk_E
$$

with $k_E = (\mathbf{k},k_D)\,,\ z_E = (\mathbf{z},z_D)\,$. The spherical polar coordinates of k_E are $k,\phi,\theta_1,\theta_2,\ldots,\theta_{D-2}$ and we have

$$
\left\{
\begin{array}{c}
k_1 = k\cos\theta_1 \\
k_2 = k\sin\theta_1\cos\theta_2 \\
k_3 = k\sin\theta_1\sin\theta_2\cos\theta_3 \\
\cdots\cdots\cdots \\
k_{D-1} = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\cos\phi \\
k_D = k\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\sin\phi
\end{array}
\right.
$$

with $0\le\theta_i\le\pi$ for $i=1,2,\ldots,D-2$ and $0\le\phi\le 2\pi$ while $k = |k_E| = (\mathbf{k}^2+k_D^2)^{1/2}\ge 0\,$. It turns out that

$$
\frac{\partial\,(k_1,k_2,\cdots,k_D)}{\partial\,(k,\phi,\theta_1,\cdots,\theta_{D-2})} = k^{D-1}(\sin\theta_1)^{D-2}(\sin\theta_2)^{D-3}\cdots(\sin\theta_{D-2})
$$

If we choose the euclidean momentum Ok_1 axis along z_E we evidently obtain $k_E \cdot x_E = kz_E \cos\theta_1 \equiv kz_E \cos\theta$ and thereby we immediately obtain

$$\begin{aligned} I_n^D(z) &= (-1)^n\,(2\pi)^{-D} \int_0^\infty \mathrm{d}k\; k^{D-1}\,(k^2+m^2)^{-n} \\ &\times \int_0^\pi d\theta\;(\sin\theta)^{D-2}\;\exp\{ikz_E\cos\theta\} \\ &\times (2\pi) \prod_{j=2}^{D-2} \int_0^\pi d\theta_j\;(\sin\theta_j)^{D-j-1} \end{aligned}$$

Now we have

$$\begin{aligned} \int_0^\pi d\theta_j\;(\sin\theta_j)^{D-j-1} &= 2\int_0^1 \mathrm{d}t_j\;(1-t_j^2)^{(D-j-2)/2} \\ &= \int_0^1 \mathrm{d}y\; y^{-1/2}(1-y)^{(D-j)/2-1} \\ &= B(1/2, D/2-j/2) \\ &= \sqrt{\pi}\;\frac{\Gamma(D/2-j/2)}{\Gamma(D/2-j/2+1/2)} \end{aligned}$$

so that

$$\begin{aligned} &\prod_{j=2}^{D-2} \int_0^\pi d\theta_j\;(\sin\theta_j)^{D-j-1} \\ &= \frac{\pi^{(D-3)/2}\,\Gamma(1)\Gamma(3/2)\Gamma(2)\cdots\Gamma(D/2-1)}{\Gamma(3/2)\Gamma(2)\cdots\Gamma(D/2-1)\Gamma(D/2-1/2)} \\ &= \frac{\pi^{(D-3)/2}}{\Gamma(D/2-1/2)} \end{aligned}$$

and thereby

$$\begin{aligned} I_n^D(z) &= \frac{2\,(-1)^n\,(4\pi)^{-D/2}}{\Gamma(D/2-1/2)\sqrt{\pi}} \int_0^\infty \mathrm{d}k\; k^{D-1}\,(k^2+m^2)^{-n} \\ &\times \int_0^\pi d\theta\;(\sin\theta)^{D-2}\;\exp\{ikz_E\cos\theta\} \end{aligned}$$

Next we find

$$\int_0^\pi d\theta\;(\sin\theta)^{D-2}\;\exp\{ikz_E\cos\theta\}$$

$$
\begin{aligned}
&= 2\int_0^{\pi/2} d\theta\ (\sin\theta)^{D-2}\ \cos(kz_E\cos\theta) \\
&= 2\int_0^{1} (1-t^2)^{(D-3)/2}\ \cos(tkz_E)\ \mathrm{d}t
\end{aligned}
$$

The value of the latter integral is reported in [22] eq. **3.771**7. p. 464 and turns out to be

$$
\sqrt{\pi}\ \left(\frac{2}{kz_E}\right)^{D/2-1} \Gamma\left(\frac{D-1}{2}\right)\ J_{D/2-1}(kz_E) \qquad (\Re\mathrm{e}\, D > 1)
$$

so that we further obtain

$$
\begin{aligned}
I_n^D(z) &= (-1)^n\,(4\pi)^{-\,D/2}\,2^{D/2}\,z_E^{-D/2+1} \\
&\times \int_0^\infty k^{\,D/2}\,(k^2+m^2)^{-\,n}\,J_{D/2-1}(kz_E)\ \mathrm{d}k
\end{aligned}
$$

and from [22] eq. **6.565**4. p. 710 we come to the expression

$$
\begin{aligned}
I_n^D(z) &= (-1)^n\,(4\pi)^{-\,D/2}\,2^{\,D/2}\,z_E^{-D/2+1} \\
&\times \frac{m^{\,D/2-n}\,z_E^{n-1}}{2^{n-1}\Gamma(n)}\ K_{D/2-n}(mz_E) \\
&= \frac{2\,(-1)^n}{(4\pi)^{\,D/2}\,\Gamma(n)}\left(\frac{2m}{z_E}\right)^{D/2-n} K_{D/2-n}(mz_E)
\end{aligned}
$$

with $0 < \Re\mathrm{e}\, D < 4n-1\,$.

Another much more quick method to get the same result is in terms of the Mellin's transform

$$
\begin{aligned}
I_n^D(z) &= \frac{(-1)^n}{\Gamma(n)\,(2\pi)^D}\int d^{\mathrm{d}}k_E\ \exp\{\,i\,k_{E\mu}\,z_{E\mu}\} \\
&\times \int_0^\infty \mathrm{d}t\ t^{\,n-1}\exp\{-t\,k_E^2 - t\,m^2\} \\
&= \frac{(-1)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\ t^{\,n-1}\ \exp\{-\,t\,m^2\} \\
&\times \frac{1}{(2\pi)^D}\int d^{\mathrm{d}}k_E\ \exp\left\{-\,t\left(k_E - i\,\frac{z_E}{2t}\right)^2 - \frac{z_E^2}{4t}\right\} \\
&= \frac{(-1)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\int_0^\infty \mathrm{d}t\ t^{\,n-1-D/2}\ \exp\{-\,t\,m^2 - z_E^2/4t\} \\
&= \frac{2\,(-1)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\left(\frac{2m}{\sqrt{-\,\lambda}}\right)^{D/2-n} K_{\,D/2-n}\left(m\sqrt{-\,\lambda}\right)
\end{aligned}
$$

where $z_E = (\mathbf{z}^2 + z_4^2)^{1/2} = (\mathbf{z}^2 - z_0^2)^{1/2} = (-\lambda)^{1/2}$, $z^2 < 0$. In the case $n = 1$, $D = 4$ we recover the Feynman propagator outside the light-cone

$$D_F(z) = -\, I_1^4(z_E) = \frac{m}{4\pi^2\sqrt{-\lambda}}\, K_1\left(m\,\sqrt{-\,\lambda}\right) \qquad (z^2 < 0)$$

and from the series representation of the Basset function of order one

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left[\ln\frac{z}{2} + \frac{1}{2}\,\mathbf{C} - \frac{1}{2}\,\psi(2)\right] + O(z^3 \ln z)$$

we obtain the leading behaviour of the causal Green's function in the four dimensional Minkowski spacetime near the outer surface of the light-cone: namely,

$$D_F(z) \approx \frac{-\,1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\,\ln\,(m\,|\,\lambda\,|^{1/2}) \qquad (\lambda = z^2 < 0)$$

On the other side of the light-cone, *i.e.* for $z^2 > 0$, we have to use the integral representation

$$\left(\frac{1}{k^2 - m^2 + i\varepsilon}\right)^n = \frac{(-\,i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{i\,t\,(k^2 - m^2 + i\varepsilon)\right\}$$

so that

$$\begin{aligned}
I_n^D(z) &= \frac{1}{(2\pi)^D}\int d^{\mathrm{d}}k\;\;\exp\{-\,i\,k_\mu\,z^\mu\} \\
&\times\; \frac{(-\,i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{i\,t\,(k^2 - m^2 + i0)\right\} \\
&= \frac{(-\,i)^n}{\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-1}\;\exp\left\{-\,i\,m^2\left(t + \frac{z^2}{4m^2\,t}\right)\right\} \\
&\times\; \frac{1}{(2\pi)^D}\int d^{\mathrm{d}}k\;\exp\left\{i\,t\left(k - \frac{z}{2t}\right)^2\right\} \\
&= \frac{(-\,i\,)^{\,n}}{(4\pi)^{D/2}\,\Gamma(n)}\int_0^\infty \mathrm{d}t\; t^{\,n-D/2-1}\;\exp\left\{-\,i\,m^2\left(t + \frac{\lambda}{4m^2\,t}\right)\right\}
\end{aligned}$$

with $\lambda = z^2 > 0$. Now we have [22] formula **3.471**11. p. 384

$$\int_0^\infty \mathrm{d}t\; t^{\,\nu-1}\;\exp\left\{\frac{1}{2}\,i\mu\left(t + \frac{\beta^2}{t}\right)\right\} = \pi\,i\,\exp\left\{-\,\frac{1}{2}\,\pi\,i\nu\right\}\,\beta^{\,\nu}\,H_{-\nu}^{(1)}(\beta\mu)$$

with $\Im\mathrm{m}\,\mu > 0\,,\ \Im\mathrm{m}\,(\beta^2\mu) \geq 0\,.$ Hence we obtain

$$I_n^D(z) \;=\; \frac{\pi\,(-1)^{n+1}}{(4\pi)^{D/2}\,\Gamma(n)}\,\exp\left\{\frac{1}{4}\,\pi i D\right\}\left(\frac{2m}{\sqrt{\lambda}}\right)^{D/2-n} H^{(2)}_{D/2-n}(m\sqrt{\lambda})$$

For $n=1\,,\ D=4$ we recover the Feynman propagator with a time-like argument

$$D_F(z) = \frac{m}{8\pi\sqrt{\lambda}}\,\left[\,N_1(m\sqrt{\lambda}) + iJ_1(m\sqrt{\lambda})\,\right] \qquad (z^2 = \lambda > 0)$$

in agreement with the very last expression of Problem 2. From the series representations of the Bessel functions we obtain the leading behaviours

$$\begin{aligned} J_1(z) &= \frac{z}{2}\left[1 + \frac{z^2}{8} + O(z^4)\right] \\ N_1(z) &= -\,\frac{2}{\pi z} + \frac{z}{\pi}\left(\ln\frac{z}{2} + \mathbf{C}\right) + O(z) \end{aligned} \tag{3.124}$$

whence

$$D_F(z) \approx -\,\frac{1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\,\ln\left(m\,|\,\lambda\,|^{1/2}\right) \qquad (\lambda = z^2 > 0)$$

Finally, consider the causal Green's function in the four dimensional Minkowski spacetime, that means $n=1\,,\ D=4\,.$ To this concern it is convenient to set

$$z \equiv |\,\mathbf{z}\,| \qquad k = |\,\mathbf{k}\,| \qquad \mathbf{k}\cdot\mathbf{z} = kz\cos\theta$$

and thereby

$$D_F(z^0,z) = \frac{i}{z\,(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{d}k\;k\,\sin(kz)\int_{-\infty}^{\infty}\mathrm{d}k_0\;\frac{\exp\{-\,iz^0k_0\}}{k_0^2 - k^2 - m^2 + i\varepsilon}$$

The last integral has two simple poles in the complex energy plane

$$k_0 = \omega(k) - i\varepsilon \qquad k_0 = -\,\omega(k) + i\varepsilon \qquad \omega(k) = \sqrt{k^2+m^2}$$

For $z^0 > 0$ we have to close the contour in the lower half-plane of the complex energy, that yields

$$\begin{aligned} D_F(z^0,z) \;&=\; \frac{\theta(z^0)}{iz\,(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}k\;k\,\exp\{ikz\}\,\frac{\exp\{-\,i(z^0 - i0)\omega(k)\}}{2\omega(k)} \\ &=\; \frac{\theta(z^0)}{8iz\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k\;k}{\sqrt{k^2+m^2}}\,\exp\left\{ikz -\, i(z_0 - i0)\sqrt{k^2+m^2}\,\right\} \end{aligned}$$

Conversely, for $z^0 < 0$ we have to close the contour in the upper half-plane $\Im\mathrm{m}\,(k_0) > 0$ that gives

$$\begin{aligned} D_F(z^0, z) &= \frac{\theta(-z^0)}{8iz\pi^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k\ k}{\sqrt{k^2+m^2}} \\ &\times\ \exp\left\{-ikz + i(z_0 + i0)\sqrt{k^2+m^2}\,\right\} \end{aligned}$$

As a consequence, for $z^0 = 0$ we obtain

$$\begin{aligned} D_F(0, z) &= \frac{1}{8z\pi^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k\ k\ \sin(kz)}{\sqrt{k^2+m^2}} \\ &= \frac{1}{4z\pi^2} \left(-\frac{d}{dz}\right) \int_{0}^{\infty} \frac{\mathrm{d}k\ \cos(kz)}{\sqrt{k^2+m^2}} \\ &= \frac{m}{4z\pi^2}\, K_1(mz) \end{aligned}$$

and thanks to Lorentz covariance

$$D_F(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\ K_1\left(m\sqrt{-z^2}\right) \qquad (z^2 < 0)$$

in accordance with the previously obtained result. Finally, when $m = 0$ we find

$$\begin{aligned} \lim_{m\to 0}\ D_F(z^0, z) &= \frac{1}{8iz\pi^2}\ \times \\ \int_{-\infty}^{\infty} \mathrm{d}k\ &\left[\,\theta(z^0)\exp\{ik(z - z^0)\} + \theta(-z^0)\exp\{-ik(z - z^0)\}\,\right] \\ = \frac{1}{4\pi i}\,\delta(z_0^2 - z^2) &= \frac{1}{4\pi i}\,\delta(\lambda) \end{aligned}$$

and consequently we eventually come to the singular behaviour in the neighbourhood of the light-cone

$$D_F(z) \approx \frac{1}{4\pi i}\,\delta(\lambda) - \frac{1}{4\pi^2\lambda} + \frac{m^2}{8\pi^2}\ \ln\left(m\,|\,\lambda\,|^{1/2}\right) \qquad (z^2 = \lambda\,\sim\,0)$$

Chapter 4

The Spinor Field

4.1 The Dirac Equation

We have already obtained the Poincaré invariant and parity-even kinetic term (2.61) for the Dirac wave field as well as the other parity-even local invariant (2.60) quadratic in the Dirac spinor fields. Then, it is easy to set up the most general Lagrange density for the free Dirac field, which satisfies the general requirements listed in Sect. 2.2 : namely,

$$\mathcal{L}_D = \tfrac{1}{2}\,\bar{\psi}\,(x)\,\gamma^{\mu}\,\mathrm{i}\,\overleftrightarrow{\partial}_{\mu}\psi\,(x) - M\,\bar{\psi}\,(x)\,\psi\,(x) \tag{4.1}$$

Beside this form of the Dirac lagrangian, in which the kinetic term contains the left-right derivative operator $\overleftrightarrow{\partial}$, we can also use the equivalent form, up to a four divergence term,

$$\begin{aligned} \bar{\mathcal{L}}_D &= \bar{\psi}\,(x)\,\gamma^{\mu}\,\mathrm{i}\,\partial_{\mu}\psi\,(x) - M\,\bar{\psi}\,(x)\,\psi\,(x) \\ &\doteq \mathcal{L}_D - \tfrac{1}{2}\,\mathrm{i}\,\partial_{\mu}\left(\,\bar{\psi}\,(x)\,\gamma^{\mu}\,\psi\,(x)\,\right) \end{aligned} \tag{4.2}$$

Notice that the spinor fields in the four dimensional Minkowski spacetime have canonical dimensions in natural units $[\,\psi\,] = \mathrm{cm}^{-3/2} = \mathrm{eV}^{\,3/2}$. The free spinor wave equation can be obtained as the Euler-Lagrange field equation from the above lagrangian by treating $\psi(x)$ and $\bar{\psi}\,(x)$ as independent fields. This actually corresponds to take independent variations with respect to $\Re\mathrm{e}\,\psi_{\alpha}(x)$ and $\Im\mathrm{m}\,\psi_{\beta}(x)$, in which the spinor component indices run over the values $\alpha, \beta = 1L, 2L, 1R, 2R$. Then we obtain the celebrated *Dirac equation*

$$(\mathrm{i}\,\partial\!\!\!/ - M)\,\psi(x) = 0 \tag{4.3}$$

where I have employed the customary notation

$$\mathrm{i}\,\partial\!\!\!/ \equiv \gamma^{\mu}\,\mathrm{i}\partial_{\mu}$$

Taking the hermitean conjugate of the Dirac equation

$$0 = \mathrm{i}\partial_\mu \psi^\dagger(x)\gamma^{\mu\dagger} + M\psi^\dagger(x) = \mathrm{i}\partial_\mu \psi^\dagger(x)\gamma^0\gamma^\mu\gamma^0 + M\psi^\dagger(x) \tag{4.4}$$

and after multiplication by γ^0 from the right we come to the adjoint Dirac equation

$$\mathrm{i}\partial_\mu \bar{\psi}(x)\gamma^\mu + M\bar{\psi}(x) \equiv \bar{\psi}(x)(\mathrm{i}\overleftarrow{\not\partial} + M) = 0 \tag{4.5}$$

The Dirac equation (4.3) can also be written *à la* Schrödinger in the form

$$\mathrm{i}\hbar\,\frac{\partial\psi}{\partial t} = H\psi \qquad\qquad H = \alpha^k p^k + \beta M \tag{4.6}$$

where H denotes the 1-particle hamiltonian self–adjoint operator with $\alpha^k = \gamma^0\gamma^k$, $p_k = \mathrm{i}\partial_k$. Owing to the transformation rule (2.80) it is immediate to verify the covariance of the Dirac equation, that means

$$\begin{aligned}(\mathrm{i}\not\partial' - M)\,\psi'(x') &= \left(\mathrm{i}\gamma^\mu\,\Lambda_\mu^{\ \nu}\,\partial_\nu - M\right)\Lambda_{\frac12}(\omega)\,\psi(x)\\ &= \Lambda_{\frac12}(\omega)\,(\mathrm{i}\not\partial - M)\,\psi(x) = 0\end{aligned} \tag{4.7}$$

To solve the Dirac equation, let us first consider the plane wave stationary solutions

$$\psi_p(x) = \Gamma(p)\,\exp\{-\,\mathrm{i}\,p\cdot x\} \tag{4.8}$$

where the spinor $\Gamma(p)$ fulfills the algebraic equation

$$(\not p - M)\,\Gamma(p) = 0 \quad \Leftrightarrow \quad (H - p_0)\,\Gamma(p) = 0 \tag{4.9}$$

H being the hermitean matrix (4.6), which admits nontrivial solutions iff

$$\det \parallel \not p - M \parallel = 0 \tag{4.10}$$

This determinant, which is obviously independent of the $\gamma-$matrices specific representation, can be most easily computed in the so called *ordinary or standard or even Dirac representation*, that is

$$\gamma_D^0 \equiv \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma_D^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{4.11}$$

If we choose the momentum rest frame $\mathbf{p} = 0$ we find solutions iff

$$\det \parallel \not p - M \parallel = (p_0 - M)^2(p_0 + M)^2 = 0 \tag{4.12}$$

and using Lorentz covariance

$$\det \parallel \not p - M \parallel = (p^2 - M^2)^2 = 0$$

which drives to the two pairs of degenerate solutions with frequencies

$$p^0_{\pm} \;=\; \pm \left(\mathbf{p}^2 + M^2\right)^{\frac{1}{2}} \;\equiv\; \pm\,\omega_{\mathbf{p}} \tag{4.13}$$

As a consequence, it follows that we have two couples of plane wave stationary solutions (4.8) with two possible *polarization states* $(r = 1, 2)$:

$$\psi_{p\,,\,r}\,(x) = \left\{ \begin{array}{ll} \Gamma_{-\,,\,r}\,(p)\,\mathrm{e}^{-\,\mathrm{i}px} & p_0 = \omega_{\mathbf{p}} \\ \Gamma_{+\,,\,r}\,(-\,p)\,\mathrm{e}^{\,\mathrm{i}px} & p_0 = -\,\omega_{\mathbf{p}} \end{array} \right. \qquad (\,r = 1, 2\,) \tag{4.14}$$

with

$$(\not p - M)\,\Gamma_{-\,,\,r}\,(p) = 0 \qquad (-\not p - M)\,\Gamma_{+\,,\,r}\,(-\,p) = 0 \qquad (\,p_0 = \omega_{\mathbf{p}}\,)$$

Actually, it is a well established convention to set

$$\Gamma_{-\,,\,r}\,(p) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\,u_r\,(p) = [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\,u_r\,(\mathbf{p})$$

$$\Gamma_{+\,,\,r}\,(-\,p) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\,v_r\,(p) = [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\,v_r\,(\mathbf{p})$$

together with

$$\begin{aligned} u_{\mathbf{p}\,,\,r}\,(x) &= [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1/2}\,u_r\,(\mathbf{p})\,\exp\{-\,\mathrm{i}\,t\,\omega_{\mathbf{p}} + \mathrm{i}\,\mathbf{p}\cdot\mathbf{x})\} \\ v_{\mathbf{p}\,,\,r}\,(x) &= [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1/2}\,v_r\,(\mathbf{p})\,\exp\{+\,\mathrm{i}\,t\,\omega_{\mathbf{p}} - \mathrm{i}\,\mathbf{p}\cdot\mathbf{x})\} \end{aligned}$$

The so called spin-amplitudes or spin-states do fulfill

$$(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k - M)\,u_r(\mathbf{p}) = 0 \qquad (\,r = 1, 2\,) \tag{4.15}$$

$$(\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k + M)\,v_r(\mathbf{p}) = 0 \qquad (\,r = 1, 2\,) \tag{4.16}$$

which is nothing but that the degenerate solution of the eigenvalue problem

$$H\,u_r(\mathbf{p}) = \omega_{\mathbf{p}}\,u_r(\mathbf{p}) \qquad H\,v_r(-\mathbf{p}) = -\,\omega_{\mathbf{p}}\,v_r(-\mathbf{p}) \qquad (\,r = 1, 2\,) \tag{4.17}$$

together with the orthonormality and closure relations

$$u_r^{\dagger}(\mathbf{p})\,u_s(\mathbf{p}) = \,2\omega_{\mathbf{p}}\,\delta_{rs} = \,v_r^{\dagger}(\mathbf{p})\,v_s(\mathbf{p}) \tag{4.18}$$

$$\sum_{r=1,2}\,\left[\,u_r(\mathbf{p})\otimes u_r^{\dagger}(\mathbf{p}) + v_r(-\mathbf{p})\otimes v_r^{\dagger}(-\mathbf{p})\,\right] = 2\omega_{\mathbf{p}} \tag{4.19}$$

The conventional normalization (4.18) is fixed in order to recover

$$\int \mathrm{d}\mathbf{x}\; u^{\dagger}_{\mathbf{q}\,,\,s}(x)\, u_{\mathbf{p}\,,\,r}(x) = \delta_{rs}\, \delta(\mathbf{p}-\mathbf{q}) = \int \mathrm{d}\mathbf{x}\; v^{\dagger}_{\mathbf{q}\,,\,s}(x)\, v_{\mathbf{p}\,,\,r}(x) \qquad (4.20)$$

while the closure relation (4.19) indeed ensures the completeness relation

$$\sum_{\mathbf{p}\,,\,r} \Big(u_{\mathbf{p}\,,\,r}(x) \otimes u^{\dagger}_{\mathbf{p}\,,\,r}(y) + v_{\mathbf{p}\,,\,r}(x) \otimes v^{\dagger}_{\mathbf{p}\,,\,r}(y) \Big)_{x_0\,=\,y_0} \;=\; \delta(\mathbf{x}-\mathbf{y})$$

in which the notation has been introduced for brevity

$$\sum_{\mathbf{p}\,,\,r} \stackrel{\mathrm{def}}{=} \sum_{r=1,2} \int \mathrm{d}\mathbf{p}$$

It is worthwhile to notice that, from the covariance (4.7) of the Dirac equation, the transformation property of the spin-states readily follows. For instance, from eq. (2.80) we find

$$\begin{aligned}
(\not p' - M)\, u_r(p') \;&=\; \Big(\Lambda_{\mu}^{\;\;\nu}\, p_{\nu}\, \gamma^{\mu} - M\Big)\, u_r(\Lambda p) \\
&=\; \Big(\Lambda_{\mu}^{\;\;\nu}\, p_{\nu}\, \Lambda^{\mu}_{\;\;\rho}\, \Lambda_{\frac12}\, \gamma^{\rho}\, \Lambda^{-1}_{\frac12} - M\Big)\, u_r(\Lambda p) \\
&=\; \Lambda_{\frac12}\, (\not p - M)\, \Lambda^{-1}_{\frac12}\, u_r(\Lambda p) \\
&=\; \Lambda_{\frac12}\, (\not p - M)\, u_r(p) \;=\; 0 \qquad (4.21)
\end{aligned}$$

Hence, the Lorentz covariance of the Dirac equation actually occurs, provided the following relationships hold true for the spin-states

$$u_r(p') = u_r(\Lambda p) = \Lambda_{\frac12}\, u_r(p) \;\Leftrightarrow\; u_r(p) = \Lambda^{-1}_{\frac12}\, u_r(\Lambda p) \qquad (4.22)$$

$$v_r(p') = v_r(\Lambda p) = \Lambda_{\frac12}\, v_r(p) \;\Leftrightarrow\; v_r(p) = \Lambda^{-1}_{\frac12}\, v_r(\Lambda p) \qquad (4.23)$$

It is not difficult to prove the further equalities : namely,

$$\bar u_r(\mathbf{p})\, u_s(\mathbf{p}) = \; 2M\, \delta_{rs} \; = -\, \bar v_r(\mathbf{p})\, v_s(\mathbf{p}) \qquad (4.24)$$

$$\begin{aligned}
&u^{\dagger}_r(\mathbf{p})\, v_s(-\mathbf{p}) = 0 \qquad \bar u_r(\mathbf{p})\, v_s(\mathbf{p}) = 0 \\
&v^{\dagger}_r(\mathbf{p})\, u_s(-\mathbf{p}) = 0 \qquad \bar v_r(\mathbf{p})\, u_s(\mathbf{p}) = 0 \qquad (r, s = 1, 2) \qquad (4.25)
\end{aligned}$$

`Proof`. From eq. (4.15) we obviously get

$$u^{\dagger}_s(\mathbf{p})\, (\omega_{\mathbf{p}}\gamma^0 - \gamma^k p^k - M)\, u_r(\mathbf{p}) = 0 \qquad (\forall\, r, s = 1, 2)$$

and taking the adjoint equation

$$u_s^\dagger(\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0+\gamma^k p^k-M)\,u_r(\mathbf{p})=0 \qquad (\forall\, r,s=1,2)$$

so that adding together

$$\bar{u}_s(\mathbf{p})\,u_r(\mathbf{p})=\frac{M}{\omega_{\mathbf{p}}}\,u_s^\dagger(\mathbf{p})\,u_r(\mathbf{p})=2M\,\delta_{rs}$$

In a quite analogous way one can readily prove that

$$\bar{v}_s(\mathbf{p})\,v_r(\mathbf{p}) \;=\; -\,\frac{M}{\omega_{\mathbf{p}}}\,v_s^\dagger(\mathbf{p})\,v_r(\mathbf{p})=-\,2M\,\delta_{rs}$$

Moreover, from eq.s (4.15) and (4.16) we obtain

$$\bar{v}_s(-\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0-\gamma^k p^k-M)\,u_r(\mathbf{p})=0 \qquad (\forall r,s=1,2)$$
$$\bar{u}_s(\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0+\gamma^k p^k+M)\,v_r(-\mathbf{p})=0 \qquad (\forall r,s=1,2)$$

and taking the hermitean conjugated of the very last equality

$$\bar{v}_s(-\mathbf{p})\,(\omega_{\mathbf{p}}\gamma^0+\gamma^k p^k+M)\,u_r(\mathbf{p})=0 \qquad (\forall r,s=1,2)$$

so that, by summing up the first equality, we get

$$v_s^\dagger(-\mathbf{p})\,u_r(\mathbf{p})=0 \qquad (\forall r,s=1,2)$$

and analogously

$$u_s^\dagger(\mathbf{p})\,v_r(-\mathbf{p})=0 \qquad (\forall r,s=1,2)$$

which completes the proof. *Quod Erat Demonstrandum*

Notice that from the above orthogonality properties of the spin-states the following orthogonality relations hold true between the positive and negative frequency eigenspinor wave functions

$$\int \mathrm{d}\mathbf{x}\; u^\dagger_{\mathbf{q}\,,\,s}(x)\,v_{\mathbf{p}\,,\,r}(x)=0=\int \mathrm{d}\mathbf{x}\; v^\dagger_{\mathbf{q}\,,\,s}(x)\,u_{\mathbf{p}\,,\,r}(x) \tag{4.26}$$

It follows therefrom that the most general solution of the Dirac equation can be written in the form

$$\psi(x) \;=\; \sum_{\mathbf{p}\,,\,r}\Big[\,c_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}(x)+d^*_{\mathbf{p}\,,\,r}\,v_{\mathbf{p}\,,\,r}(x)\Big] \tag{4.27}$$
$$\bar{\psi}(x) \;=\; \sum_{\mathbf{p}\,,\,r}\Big[\,c^*_{\mathbf{p}\,,\,r}\,\bar{u}_{\mathbf{p}\,,\,r}(x)+d_{\mathbf{p}\,,\,r}\,\bar{v}_{\mathbf{p}\,,\,r}(x)\Big] \tag{4.28}$$

which is nothing but the *normal mode expansion* of the free Dirac spinor classical wave field, where $c_{\mathbf{p}\,,\,r}$ and $d_{\mathbf{p}\,,\,r}$ are arbitrary complex coefficients.

In the chiral representation (2.66) for the gamma matrices we can build up a very convenient set of spin-states as follows. Consider the matrices

$$\not p \pm M = \begin{pmatrix} \pm M & 0 & \omega_{\mathbf{p}} - p_z & -p_x + \mathrm{i}\, p_y \\ 0 & \pm M & -p_x - \mathrm{i}\, p_y & \omega_{\mathbf{p}} + p_z \\ \omega_{\mathbf{p}} + p_z & p_x - \mathrm{i}\, p_y & \pm M & 0 \\ p_x + \mathrm{i}\, p_y & \omega_{\mathbf{p}} - p_z & 0 & \pm M \end{pmatrix} \tag{4.29}$$

where we have set $\mathbf{p} = (p^1, p^2, p^2) = (p_x, p_y, p_z)$. Notice that we can define the two projectors on the bidimensional spaces spanned by the positive energy and negative energy spin-states respectively : namely,

$$\mathcal{E}_{\pm}(p) \;\equiv\; (M \pm \not p)\,/\,2M \qquad\qquad (\,p_0 = \omega_{\mathbf{p}}\,) \tag{4.30}$$

which satisfy by definition

$$\begin{gathered} \mathcal{E}_{\pm}^2 = \mathcal{E}_{\pm} \qquad \mathcal{E}_{+}\,\mathcal{E}_{-} = 0 = \mathcal{E}_{-}\,\mathcal{E}_{+} \\ \operatorname{tr}\mathcal{E}_{\pm} = 2 \qquad \mathcal{E}_{+} + \mathcal{E}_{-} = \mathbf{I} \end{gathered}$$

Moreover we have

$$\mathcal{E}_{\pm}^{\dagger}(p) \;=\; \mathcal{E}_{\pm}(\tilde{p}) \qquad\qquad \tilde{p}^{\,\mu} = p_{\mu} \tag{4.31}$$

Then, if we introduce the constant bispinors

$$\xi_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \eta_1 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \eta_2 \equiv \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which are the eigenvectors of the γ^0 matrix

$$\gamma^0\,\xi_r = \xi_r \qquad \gamma^0\,\eta_r = -\,\eta_r \qquad (\,r = 1, 2\,)$$

and do indeed satisfy by direct inspection

$$\xi_r^{\top}\,\gamma^k\,\xi_s = 0 = \eta_r^{\top}\,\gamma^k\,\eta_s \qquad\qquad \forall\, r, s = 1, 2 \,\vee\, k = 1, 2, 3$$

then we can suitably define

$$\begin{cases} u_r\,(\mathbf{p}) \equiv 2M\,(2\omega_{\mathbf{p}} + 2M)^{-1/2}\;\mathcal{E}_{+}\,\xi_r \\ v_r\,(\mathbf{p}) \equiv 2M\,(2\omega_{\mathbf{p}} + 2M)^{-1/2}\;\mathcal{E}_{-}\,\eta_r \end{cases} \qquad (\,r = 1, 2\,) \tag{4.32}$$

the explicit form of which is given by

$$u_1(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} M + \omega_{\mathbf{p}} - p_z \\ -p_x - \mathrm{i}\,p_y \\ M + \omega_{\mathbf{p}} + p_z \\ p_x + \mathrm{i}\,p_y \end{pmatrix}$$

$$u_2(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -p_x + \mathrm{i}\,p_y \\ M + \omega_{\mathbf{p}} + p_z \\ p_x - \mathrm{i}\,p_y \\ M + \omega_{\mathbf{p}} - p_z \end{pmatrix}$$

$$v_1(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -p_x + \mathrm{i}\,p_y \\ M + \omega_{\mathbf{p}} + p_z \\ -p_x + \mathrm{i}\,p_y \\ -M + p_z - \omega_{\mathbf{p}} \end{pmatrix}$$

$$v_2(\mathbf{p}) = (2\omega_{\mathbf{p}} + 2M)^{-1/2} \begin{pmatrix} -M - \omega_{\mathbf{p}} + p_z \\ p_x + \mathrm{i}\,p_y \\ \omega_{\mathbf{p}} + M + p_z \\ p_x + \mathrm{i}\,p_y \end{pmatrix} \tag{4.33}$$

their orthonormality and completeness relations being in full accordance with formulæ (4.18), (4.19) and (4.25). In fact we have for instance

$$\begin{aligned} v_r^{\dagger}(\mathbf{p})\, v_s(\mathbf{p}) &= (2\omega_{\mathbf{p}} + 2M)^{-1}\, \eta_r^{\top} (M - \tilde{\not{p}})\, (M - \not{p}) \eta_s \\ &= (2\omega_{\mathbf{p}} + 2M)^{-1}\, \eta_r^{\top} \left(M^2 - 2M\gamma^0 \omega_{\mathbf{p}} + \omega_{\mathbf{p}}^2 + \mathbf{p}^2 \right) \eta_s \\ &= (2\omega_{\mathbf{p}} + 2M)^{-1}\, \eta_r^{\top} \left(2\omega_{\mathbf{p}}^2 + 2M\omega_{\mathbf{p}} \right) \eta_s \\ &= 2\omega_{\mathbf{p}}\, \tfrac{1}{2}\, \eta_r^{\top} \eta_s = 2\omega_{\mathbf{p}}\, \delta_{rs} \end{aligned}$$

in which I have made use of the property

$$\eta_r^{\top} \gamma^k \gamma^0 \eta_s = -\, \eta_r^{\top} \gamma^k \eta_s = 0 \qquad \forall\, r, s = 1, 2 \ \vee\ k = 1, 2, 3$$

Finally, taking the normalization (4.24) into account together with

$$\mathcal{E}_+\ u_r(\mathbf{p}) = u_r(\mathbf{p}) \qquad \mathcal{E}_-\ v_r(\mathbf{p}) = v_r(\mathbf{p}) \qquad (r = 1, 2)$$

it is immediate to obtain the so called sums over the spin-states, that is

$$\sum_{r=1,\,2} \begin{cases} u_r(\mathbf{p}) \otimes \bar{u}_r(\mathbf{p}) = 2M\mathcal{E}_+(p) \\ v_r(\mathbf{p}) \otimes \bar{v}_r(\mathbf{p}) = -\,2M\mathcal{E}_-(p) \end{cases} \tag{4.34}$$

or even for $\alpha, \beta = 1L, 2L, 1R, 2R$,

$$\sum_{r=1,\,2} \begin{cases} u_{r\,,\,\alpha}(\mathbf{p})\, \bar{u}_{r\,,\,\beta}(\mathbf{p}) = (\not{p} + M\,)_{\alpha\beta} \\ v_{r\,,\,\alpha}(\mathbf{p})\, \bar{v}_{r\,,\,\beta}(\mathbf{p}) = (\not{p} - M\,)_{\alpha\beta} \end{cases} \qquad (\,p_0 = \omega_{\mathbf{p}}\,) \tag{4.35}$$

Hence, from the orthonormality relations (4.24) we can immediately verify that we have

$$\sum_{r=1,\,2} u_r\,(\mathbf{p}) \otimes \bar{u}_r\,(\mathbf{p})\, u_s\,(\mathbf{p}) = 2M\, u_s\,(\mathbf{p}) = 2M\mathcal{E}_+\, u_s\,(\mathbf{p})$$
$$\sum_{r=1,\,2} v_r\,(\mathbf{p}) \otimes \bar{v}_r\,(\mathbf{p})\, v_s\,(\mathbf{p}) = -\,2M\, v_s\,(\mathbf{p}) = -\,2M\mathcal{E}_-\, v_s\,(\mathbf{p})$$

4.2 Noether Currents

From Noether theorem and from the Lagrange density (4.1) we obtain the canonical energy momentum tensor of the free Dirac spinor wave field which turns out to be real though not symmetric

$$\begin{aligned} T^{\mu}{}_{\nu}(x) &\equiv (\delta\,\mathcal{L}_D\,/\,\delta\,\partial_\mu\,\psi)\,\partial_\nu\,\psi + \partial_\nu\,\bar\psi\,(\delta\,\mathcal{L}_D\,/\,\delta\,\partial_\mu\,\bar\psi) - \delta^{\mu}_{\nu}\,\mathcal{L}_D \\ &= \frac{\mathrm{i}}{2}\Big(\bar\psi(x)\,\gamma^{\mu}\,\partial_\nu\,\psi(x) - \partial_\nu\,\bar\psi(x)\,\gamma^{\mu}\,\psi(x)\Big) \end{aligned} \tag{4.36}$$

$$T_{\mu\nu}(x) \neq T_{\nu\mu}(x)$$

where we have taken into account that the Dirac lagrangian vanishes if the equations of motion hold true as it occurs in the Noether theorem. The corresponding canonical total angular momentum density tensor for the Dirac field can be obtained from the general expression (2.98) and reads

$$\begin{aligned} M^{\mu\kappa\lambda}(x) &\stackrel{\text{def}}{=} x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) + S^{\mu\kappa\lambda}(x) \\ &= x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) + \tfrac{1}{2}\,\bar\psi(x)\,\{\sigma^{\lambda\kappa}\,,\,\gamma^{\mu}\}\,\psi(x) \end{aligned}$$

As a matter of fact we have

$$\begin{aligned} \delta\,\mathcal{L}\,/\,\delta\,\partial_\mu\,\psi(x) &= \tfrac{1}{2}\,\bar\psi(x)\,\mathrm{i}\,\gamma^{\mu} & \qquad \delta\,\mathcal{L}\,/\,\delta\,\partial_\mu\,\bar\psi(x) &= -\,\tfrac{1}{2}\,\mathrm{i}\,\gamma^{\mu}\,\psi(x) \\ (-\,\mathrm{i})(\,S^{\lambda\kappa})\,\psi(x) &= -\,\mathrm{i}\,\sigma^{\lambda\kappa}\,\psi(x) & \qquad \mathrm{i}\,\bar\psi(x)\,(\,S^{\lambda\kappa}\,) &= \mathrm{i}\,\bar\psi(x)\,\sigma^{\lambda\kappa} \end{aligned}$$

where

$$\sigma^{\lambda\kappa} \equiv \frac{\mathrm{i}}{4}\,\big[\,\gamma^{\lambda}\,,\,\gamma^{\kappa}\,\big] \qquad\qquad \big(\,\sigma^{\lambda\kappa}\,\big)^{\dagger} = \gamma^{0}\,\sigma^{\lambda\kappa}\,\gamma^{0}$$

is the spin tensor for the Dirac field. Hence from the general expression (2.98) we get

$$\begin{aligned} S^{\mu\lambda\kappa}(x) &\stackrel{\text{def}}{=} \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\,u_A(x)}\,(-\,\mathrm{i})\,\big(S^{\lambda\kappa}\big)_{AB}\,u_B(x) \\ &= \delta\mathcal{L}/\delta\,\partial_\mu\,\psi(x)\,(-\,\mathrm{i})\,\big(S^{\lambda\kappa}\big)\,\psi(x) \\ &+ \mathrm{i}\,\bar\psi(x)\,\big(S^{\lambda\kappa}\big)\,\delta\mathcal{L}/\delta\,\partial_\mu\,\bar\psi(x) \\ &= \tfrac{1}{2}\,\bar\psi(x)\,\{\,\gamma^{\mu}\,,\,\sigma^{\lambda\kappa}\,\}\,\psi(x) \end{aligned}$$

Notice that

$$M^{0jk}(x) = x^{j}\,T^{0k}(x) - x^{k}\,T^{0j}(x) + \psi^{\dagger}(x)\,\sigma^{jk}\,\psi(x) \tag{4.37}$$

$$M^{0k0}(x) = x^{k}\,T^{00}(x) - x^{0}\,T^{0k}(x) \tag{4.38}$$

It is rather easy to check, using the Dirac equation, that the continuity equations actually hold true

$$\partial_\mu T^{\mu\nu} = 0 \qquad\qquad \partial_\mu M^{\mu\lambda\kappa} = 0 \tag{4.39}$$

which lead to the four conserved energy momentum Noether charge

$$\begin{aligned} P_\mu &= \int \mathrm{d}\mathbf{x}\; T^{0}{}_{\mu}(x^0,\mathbf{x}) = \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\; \psi^\dagger(x)\,\mathrm{i}\overleftrightarrow{\partial}_\mu\,\psi(x) \\ &= \int \mathrm{d}\mathbf{x}\; \psi^\dagger(x)\,\mathrm{i}\partial_\mu\,\psi(x) \end{aligned} \tag{4.40}$$

while from the spatial integration of eq. (4.37) we obtain the three conserved Noether charges corresponding to the spatial components of the relativistic total angular momentum

$$\begin{aligned} M_{jk} &= \int \mathrm{d}\mathbf{x}\, M^{0}_{\;jk}(t,\mathbf{x}) \doteq \\ &\int \mathrm{d}\mathbf{x}\left[\,x_j\,\psi^\dagger(x)\,\mathrm{i}\,\partial_k\,\psi(x) - \{\,j \leftrightarrow k\,\} + \psi^\dagger(x)\,\sigma^{jk}\,\psi(x)\right] \end{aligned} \tag{4.41}$$

in which we have discarded, as customary, the boundary term

$$\tfrac{1}{2}\,\mathrm{i}\int \mathrm{d}\mathbf{x}\,\partial_k\Big(\,x_j\,\psi^\dagger(t,\mathbf{x})\,\psi(t,\mathbf{x})\Big) - \{\,j \leftrightarrow k\,\} \;=\; 0$$

Furthermore, from the spatial integration of eq. (4.38) we find

$$M^{0k} \;=\; \int \mathrm{d}\mathbf{x}\, M^{0k0}(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\; x^k\, T^{00}(t,\mathbf{x}) - x^0\, P^k \tag{4.42}$$

in such a way that the constancy in time of these latter spacetime components of the relativistic total angular momentum leads to the definition of the velocity for the *center of the energy*, *viz.*,

$$X_t^k \;\stackrel{\mathrm{def}}{=}\; M^{-1}\int \mathrm{d}\mathbf{x}\; x^k\, T^{00}(t,\mathbf{x})$$

which corresponds to the relativistic generalization of the center of mass, that means

$$\dot{M}^{0k} = 0 \;\Leftrightarrow\; P^k = M\dot{X}_t^k = \int \mathrm{d}\mathbf{x}\; x^k\, \dot{T}^{00}(t,\mathbf{x}) \tag{4.43}$$

It follows that the so called *center of momentum frame* $\mathbf{P} = 0$ just coincides with the inertial reference frame in which the center of the energy is at rest.

Notice however that owing to the lack of symmetry for the canonical energy momentum tensor of the Dirac field [1] its spin angular momentum tensor is not constant in time. Actually we find, for instance,

$$\partial_\mu S^{\mu jk}(x) = T^{jk} - T^{kj} \neq 0 \tag{4.44}$$

and consequently the corresponding Noether charges are not conserved in time so that

$$S_{ij}\left(x^0\right) = \int \mathrm{d}\mathbf{x}\ S^0{}_{ij}(t,\mathbf{x}) = \tfrac{1}{2}\,\varepsilon_{ijk}\int \mathrm{d}\mathbf{x}\ \psi^\dagger(x)\,\Sigma_k\,\psi(x) \tag{4.45}$$

where

$$\Sigma_k \ \equiv \ \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{4.46}$$

However, in the case that the spinor wave field ψ does not depend on some of the spatial coordinates $(x^1,x^2,x^3) = (x,y,z)$ it is possible to achieve that the continuity equation should hold for some of the components of the spin angular momentum density tensor and, consequently, that the corresponding Noether charges – *i.e.* its space integrals – remain constant in time. Thus, for example, if $\partial_1\psi = 0 = \partial_2\psi$ so that $\psi(t,z) = \psi(t,0,0,z)$ and cosequently we obtain

$$2\mathrm{i}\,T^{12} = \bar\psi(t,z)\,\gamma^1\,\overleftrightarrow{\partial}_y\,\psi(t,z) = 0 = \bar\psi(t,z)\,\gamma^2\,\overleftrightarrow{\partial}_x\,\psi(t,z) = 2\mathrm{i}\,T^{21}$$

and thereby

$$\partial_\mu M^\mu_{12} = \partial_\mu S^\mu_{12} = 0 \tag{4.47}$$

Hence it follows that the component of the spin vector along the direction of propagation, which is named the *helicity*, is conserved in time : namely,

$$\frac{dh}{\mathrm{d}t} \ = \ 0 \qquad h \ \stackrel{\mathrm{def}}{=} \ \int_{-\infty}^{\infty} \mathrm{d}z\ \tfrac{1}{2}\,\psi^\dagger(t,z)\,\Sigma_3\,\psi(t,z) \tag{4.48}$$

After insertion of the normal modes expansions (4.28) one gets

$$h = \tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}p\ \left[c^*_{p,1}\,c_{p,1} - c^*_{p,2}\,c_{p,2} - d_{p,1}\,d^*_{p,1} + d_{p,2}\,d^*_{p,2}\right] \tag{4.49}$$

[1]The lack of symmetry in the canonical energy momentum tensor of the Dirac field can be removed using a trick due to J. Belifante, *Physica* **6** (1939) 887, *ibidem* **7** (1940) 449, see **Problem** 1.

`Proof.` By passing in (4.46) to the momentum representations (4.28) and carrying out the integration over the three dimensional configuration space we obtain

$$\begin{aligned}
h &= \int_{-\infty}^{\infty} \mathrm{d}z\ \tfrac{1}{2}\,\psi^{\dagger}(t,z)\,\Sigma_3\,\psi(t,z) \\
&= \int_{-\infty}^{\infty} \mathrm{d}z \sum_{p\,,\,r} \left[c^{*}_{p\,,\,r}\, u^{\dagger}_{p\,,\,r}(t,z) + d_{p\,,\,r}\, v^{\dagger}_{p\,,\,r}(t,z) \right] \\
&\times\ \tfrac{1}{2}\,\Sigma_3 \sum_{q\,,\,s} \left[c_{q\,,\,s}\, u_{q\,,\,s}(t,z) + d^{*}_{q\,,\,s}\, v_{q\,,\,s}(t,z) \right]
\end{aligned}$$

in which we have set $\mathbf{p} = (0,0,p)\,, \mathbf{q} = (0,0,q)\,,\ \omega_p = \surd(p^2 + m^2)$ together with

$$\begin{aligned}
u_{p\,,\,r}(t,z) &= [\,4\pi\omega_p\,]^{-1/2}\, u_r(p)\, \exp\{-\,\mathrm{i}\,t\,\omega_p + \mathrm{i}\,pz\} \qquad (r = 1,2) \\
v_{q\,,\,s}(t,z) &= [\,4\pi\omega_q\,]^{-1/2}\, v_s(q)\, \exp\{+\,\mathrm{i}\,t\,\omega_q - \mathrm{i}\,qz\} \qquad (s = 1,2)
\end{aligned}$$

the normalization being now consistent with the plane waves independent of the transverse $x^{\top} = (x^1\,,\,x^2)$ spatial coordinates. From (4.25) and the commutation relation

$$[\,\omega_p\gamma^0 - p\,\gamma^3\,,\,\Sigma_3\,] = 0 \tag{4.50}$$

together with the definition (4.32)

$$\begin{cases} u_r(p) \equiv (2\omega_p + 2M)^{-1/2}\ \left(M + \omega_p\gamma^0 - p\,\gamma^3\right)\,\xi_r \\ v_r(p) \equiv (2\omega_p + 2M)^{-1/2}\ \left(M - \omega_p\gamma^0 + p\,\gamma^3\right)\,\eta_r \end{cases} \qquad (r = 1,2) \tag{4.51}$$

it can be readily derived that

$$\begin{gathered}
(\,\Sigma_3 - 1\,)\,u_1(p) = (\,\Sigma_3 - 1\,)\,v_2(p) = 0 \\
(\,\Sigma_3 + 1\,)\,u_2(p) = (\,\Sigma_3 + 1\,)\,v_1(p) = 0 \\
u^{\dagger}_r(p)\,\Sigma_3\,v_s(-\,q) = 0 \qquad v^{\dagger}_r(-\,p)\,\Sigma_3\,u_s(q) = 0 \qquad (r,s = 1,2)
\end{gathered} \tag{4.52}$$

Hence the spin component along the direction of propagation, which is named *helicity* of the Dirac spinor wave field, turns out to be time independent and takes the form

$$\begin{aligned}
h &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{4\omega_p} \sum_{r=1\,,\,2} \left[c^{*}_{p\,,\,r}\, c_{p\,,\,r}\, u^{\dagger}_r(p)\,\Sigma_3\,u_r(p) + d_{p\,,\,r}\, d^{*}_{p\,,\,r}\, v^{\dagger}_r(p)\,\Sigma_3\,v_r(p) \right] \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p\, \left[c^{*}_{p\,,\,1}\, c_{p\,,\,1} - c^{*}_{p\,,\,2}\, c_{p\,,\,2} - d_{p\,,\,1}\, d^{*}_{p\,,\,1} + d_{p\,,\,2}\, d^{*}_{p\,,\,2} \right]
\end{aligned}$$

which proves equation (4.49). *Quod Erat Demonstrandum*

Finally, the Dirac lagrangian is manifestly invariant under the internal symmetry group $U(1)$ of the phase transformations

$$\psi'(x) = \mathrm{e}^{\mathrm{i}q\alpha}\,\psi(x) \qquad \bar\psi'(x) = \bar\psi(x)\,\mathrm{e}^{-\,\mathrm{i}q\alpha} \qquad 0 \le \alpha < 2\pi$$

where q denotes the particle electric charge, *e.g.* $q = -\,e$ $(e > 0)$ for the electron. Then from eq. (2.101) the corresponding Noether current becomes

$$J^{\mu}(x) = q\,\bar\psi(x)\,\gamma^{\mu}\,\psi(x) \tag{4.53}$$

and will be identified with the electric current carried on by the spinor field, which transforms as a true four vector under the space inversion (2.59) or, more generally, under the improper orthochronus Lorentz transformations of $L_{-}^{\uparrow}$, that means

$$J_0^{\prime}(x^0, -\mathbf{x}) = J_0(x) \qquad \mathbf{J}^{\prime}(x^0, -\mathbf{x}) = -\mathbf{J}(x)$$

If instead we consider the internal symmetry group $U(1)$ of the *chiral* phase transformations

$$\psi^{\prime}(x) = \exp\{-\mathrm{i}\alpha\gamma_5\}\psi(x) \qquad \bar{\psi}^{\prime}(x) = \bar{\psi}(x)\exp\{-\mathrm{i}\alpha\gamma_5\} \qquad 0 \le \alpha < 2\pi$$

then the free Dirac Lagrange density is not invariant, owing to the presence of the mass term, so that the Noether theorem (2.91) yields in this case

$$\partial_\mu J_5^\mu(x) = \partial_\mu\Big(\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\Big) = 2\mathrm{i}M\bar{\psi}(x)\gamma_5\psi(x) \tag{4.54}$$

Notice that the *axial vector current* $J_5^\mu(x)$ is a pseudovector since we have the transformation law under the space inversion (2.59)

$$J_5^{0\,\prime}(x^0, -\mathbf{x}) = -J_5^0(x) \qquad J_5^{k\,\prime}(x^0, -\mathbf{x}) = J_5^k(x)$$

It follows therefrom that the chiral phase transformations are a symmetry group of the classical free Dirac theory only in the massless limit.

4.3 Quantization of a Dirac Field

The above discussion eventually leads to the energy momentum vector of the free Dirac wave field that reads

$$P_\mu = \int \mathrm{d}\mathbf{x}\, T^0{}_\mu(x^0,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\, \psi^\dagger(x)\, \mathrm{i}\partial_\mu\, \psi(x) \tag{4.55}$$

and inserting the normal mode expansions (4.28) we obtain

$$\begin{aligned} P_\mu &= \int \mathrm{d}\mathbf{x} \sum_{\mathbf{q},s} \left[c^*_{\mathbf{q},s}\, u^\dagger_{\mathbf{q},s}(x) \;+\; d_{\mathbf{q},s}\, v^\dagger_{\mathbf{q},s}(x) \right] \\ &\times \sum_{\mathbf{p},r} p_\mu \left[c_{\mathbf{p},r}\, u_{\mathbf{p},r}(x) \;-\; d^*_{\mathbf{p},r}\, v_{\mathbf{p},r}(x) \right] \end{aligned} \tag{4.56}$$

where we have set $p_\mu \equiv (\omega_{\mathbf{p}}\,,\,-\mathbf{p})$. Taking the orthonormality relations (4.20) and (4.26) into account we come to the expression

$$P^\mu = \sum_{\mathbf{p},r} p^\mu \left(c^*_{\mathbf{p},r}\, c_{\mathbf{p},r} \;-\; d_{\mathbf{p},r}\, d^*_{\mathbf{p},r} \right) \tag{4.57}$$

Now the key point: as we shall see in a while, in order to quantize the relativistic spinor wave field in such a manner to obtain a *positive semidefinite energy operator*, then we must impose canonical anticommutation relations. As a matter of fact, once the normal mode expansion coefficients turn into creation and destruction operators, that means

$$c_{\mathbf{p},r}\,,\, c^*_{\mathbf{p},r} \;\mapsto\; c_{\mathbf{p},r}\,,\, c^\dagger_{\mathbf{p},r} \qquad\qquad d_{\mathbf{p},r}\,,\, d^*_{\mathbf{p},r} \;\mapsto\; d_{\mathbf{p},r}\,,\, d^\dagger_{\mathbf{p},r}$$

had we assumed the canonical commutation relations

$$[\, d_{\mathbf{p},r}\,,\, d^\dagger_{\mathbf{q},s}\,] = \delta_{rs}\,\delta(\mathbf{p}-\mathbf{q})$$

as in the scalar field case, then we would find

$$P^\mu = \sum_{\mathbf{p},r} p^\mu \left(c^\dagger_{\mathbf{p},r}\, c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r}\, d_{\mathbf{p},r} \right) - 2\, U_0\, g^{\mu 0}$$

where U_0 is again the divergent zero-point energy (3.46)

$$\begin{aligned} cU_0 &= \delta(\mathbf{0}) \sum_{\mathbf{p}} \tfrac{1}{2}\hbar\,\omega_{\mathbf{p}} = V\hbar \int \frac{\omega_{\mathbf{p}}\,\mathrm{d}\mathbf{p}}{2(2\pi)^3} \\ &= \frac{V\hbar c}{4\pi^2} \int_0^K \mathrm{d}p\, p^2 \sqrt{p^2 + M^2c^2/\hbar^2} \end{aligned} \tag{4.58}$$

whereas V is the volume of a very large box and $\hbar K \gg Mc$ is a very large wavenumber, the factor two being due to spin. This means, however, that even assuming normal ordering prescription to discard U_0 , still the spinor energy operator P_0 is no longer positive semidefinite.

Turning back to the classical spinor wave field, it turns out that it is not convenient to understand the normal mode expansion coefficients

$$d_{\mathbf{p},r} \ \left(r = 1,2\,,\ \mathbf{p} \in \mathbb{R}^3\right)$$

as ordinary complex numbers. On the contrary, we can assume all those coefficients to be anticommuting numbers, also named *Graßmann numbers* [2]

$$\{d_{\mathbf{p},r}\,,\, d_{\mathbf{q},s}\} = 0 = \{d_{\mathbf{p},r}\,,\, d^{*}_{\mathbf{q},s}\} \qquad (\,\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q} \in \mathbb{R}^3\,)$$

in which $\{a,b\} = ab + ba$. Notice *en passant* that in the particular case $r = s$ and $\mathbf{p} = \mathbf{q}$ we find $d^2_{\mathbf{q},s} = 0 = d^{*2}_{\mathbf{q},s}$. Internal consistency then requires that also the normal mode expansion coefficients $c_{\mathbf{p},r}$ $(\,r = 1,2\,,\ \mathbf{p} \in \mathbb{R}^3\,)$ must be taken Graßmann numbers, in such a manner that the whole classical Dirac spinor relativistic wave field becomes Graßmann valued so that

$$\{\psi\,(x)\,,\,\psi\,(y)\} = \{\psi\,(x)\,,\,\bar\psi\,(y)\} = \{\bar\psi\,(x)\,,\,\bar\psi\,(y)\} = 0 \tag{4.59}$$

Under this assumption, the canonical quantization of such a system is then achieved by replacing the Graßmann numbers valued coefficients of the normal mode expansion by creation annihilation operators acting on a Fock space and postulating the *canonical anticommutation relations*, that means

$$\begin{aligned} &\{c_{\mathbf{p},r}\,,\, c^{\dagger}_{\mathbf{q},s}\} = \delta_{rs}\,\delta\,(\mathbf{p}-\mathbf{q}) = \{d_{\mathbf{p},r}\,,\, d^{\dagger}_{\mathbf{q},s}\} \\ &(\,\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q} \in \mathbb{R}^3\,) \\ &\text{all the other anticommutators vanishing} \end{aligned} \tag{4.60}$$

As a consequence, the quantized Dirac spinor wave field becomes an operator valued tempered distribution acting on a Fock space and reads

$$\psi\,(x) \;=\; \sum_{\mathbf{p},r} \left[\, c_{\mathbf{p},r}\, u_{\mathbf{p},r}\,(x) + d^{\dagger}_{\mathbf{p},r}\, v_{\mathbf{p},r}\,(x) \right] \tag{4.61}$$

$$\psi^{\dagger}(x) \;=\; \sum_{\mathbf{p},r} \left[\, c^{\dagger}_{\mathbf{p},r}\, u^{\dagger}_{\mathbf{p},r}\,(x) + d_{\mathbf{p},r}\, v^{\dagger}_{\mathbf{p},r}\,(x) \right] \tag{4.62}$$

[2] Hermann Günther Graßmann (Stettino, 15.04.1809 – 26.09.1877) introduced the modern linear algebra in his masterpiece : *Die Lineale Ausdehnungslehre, ein neuer Zweig del Mathematik (Linear Extension Theory, a New Branch of Mathematics* (1844).

The canonical anticommutation relations (4.60) actually imply

$$\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta(t,\mathbf{y})\} = 0 = \{\psi_\alpha^\dagger(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} \tag{4.63}$$

$$\{\psi_\alpha(t,\mathbf{x})\,,\,\psi_\beta^\dagger(t,\mathbf{y})\} = \delta\,(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta} \tag{4.64}$$

$$(\,\alpha,\beta = 1L, 2L, 1R, 2R\,)$$

Then, if we adopt once again the normal product to remove the divergent and negative zero-point energy contribution to the cosmological constant, we come to the operator expression for the energy momentum of the Dirac spinor quantum free field: namely,

$$\begin{aligned} P_0 &= \int \mathrm{d}\mathbf{x}\ :\,\psi^\dagger(x)\,\mathrm{i}\partial_0\,\psi(x)\,: \\ &= \int \mathrm{d}\mathbf{x}\ :\,\psi^\dagger(x)\,H\,\psi(x)\,: \\ &= \sum_{\mathbf{p}\,,\,r} \omega_{\mathbf{p}}\left[\,c^\dagger_{\mathbf{p}\,,\,r}\,c_{\mathbf{p}\,,\,r} \ +\ d^\dagger_{\mathbf{p}\,,\,r}\,d_{\mathbf{p}\,,\,r}\right] \end{aligned} \tag{4.65}$$

$$\begin{aligned} \mathbf{P} &= \int \mathrm{d}\mathbf{x}\ :\,\psi^\dagger(x)\,(-\,\mathrm{i}\,\boldsymbol{\nabla}\,)\,\psi(x)\,: \\ &= \sum_{\mathbf{p}\,,\,r} \mathbf{p}\left[\,c^\dagger_{\mathbf{p}\,,\,r}\,c_{\mathbf{p}\,,\,r} \ +\ d^\dagger_{\mathbf{p}\,,\,r}\,d_{\mathbf{p}\,,\,r}\right] \end{aligned} \tag{4.66}$$

It is important to remark that the canonical anticommutation relations

$$\{d_{\mathbf{p}\,,\,r}\,,\,d^\dagger_{\mathbf{q}\,,\,s}\} = \delta_{rs}\,\delta\,(\mathbf{p}-\mathbf{q}) \qquad \{d_{\mathbf{p}\,,\,r}\,,\,d_{\mathbf{q}\,,\,s}\} = 0$$

$$(\,\forall\, r,s = 1,2 \quad \mathbf{p},\mathbf{q}\in\mathbb{R}^3\,)$$

do guarantee the positive semidefiniteness of the energy operator P_0, while the remaining anticommutators (4.60) can be derived from the requirement that the energy momentum operators P_μ do realize the self-adjoint generators of the spacetime translations of a unitary representation of the Poincaré group. As a matter of fact

$$\begin{aligned} &[\,P_\mu\,,\,\psi(x)\,] \ = \\ &\sum_{\mathbf{p}\,,\,r} p_\mu\left[\,\Big(c^\dagger_{\mathbf{p}\,,\,r}\,c_{\mathbf{p}\,,\,r} \ +\ d^\dagger_{\mathbf{p}\,,\,r}\,d_{\mathbf{p}\,,\,r}\Big)\,,\,\sum_{\mathbf{q}\,,\,s} c_{\mathbf{q}\,,\,s}\,u_{\mathbf{q}\,,\,s}(x)\right] \ + \\ &\sum_{\mathbf{p}\,,\,r} p_\mu\left[\,\Big(c^\dagger_{\mathbf{p}\,,\,r}\,c_{\mathbf{p}\,,\,r} \ +\ d^\dagger_{\mathbf{p}\,,\,r}\,d_{\mathbf{p}\,,\,r}\Big)\,,\,\sum_{\mathbf{q}\,,\,s} d^\dagger_{\mathbf{q}\,,\,s}\,v_{\mathbf{q}\,,\,s}(x)\Big)\right] \ = \\ &\sum_{\mathbf{p}\,,\,r} p_\mu \sum_{\mathbf{q}\,,\,s} u_{\mathbf{q}\,,\,s}(x)\,\Big(c^\dagger_{\mathbf{p}\,,\,r}\,\{c_{\mathbf{p}\,,\,r}\,,\,c_{\mathbf{q}\,,\,s}\} - \{c^\dagger_{\mathbf{p}\,,\,r}\,,\,c_{\mathbf{q}\,,\,s}\}\,c_{\mathbf{p}\,,\,r}\Big) \ + \end{aligned}$$

$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} u_{\mathbf{q},s}(x)\left(d^\dagger_{\mathbf{p},r}\{d_{\mathbf{p},r}\,,\,c_{\mathbf{q},s}\} - \{d^\dagger_{\mathbf{p},r}\,,\,c_{\mathbf{q},s}\}\,d_{\mathbf{p},r}\right) +$$

$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} v_{\mathbf{q},s}(x)\left(c^\dagger_{\mathbf{p},r}\{c_{\mathbf{p},r}\,,\,d^\dagger_{\mathbf{q},s}\} - \{c^\dagger_{\mathbf{p},r}\,,\,d^\dagger_{\mathbf{q},s}\}\,c_{\mathbf{p},r}\right) +$$

$$\sum_{\mathbf{p},r} p_\mu \sum_{\mathbf{q},s} v_{\mathbf{q},s}(x)\left(d^\dagger_{\mathbf{p},r}\{d_{\mathbf{p},r}\,,\,d^\dagger_{\mathbf{q},s}\} - \{d^\dagger_{\mathbf{p},r}\,,\,d^\dagger_{\mathbf{q},s}\}\,d_{\mathbf{p},r}\right)$$

$$= \; -\,\mathrm{i}\,\partial_\mu\psi(x) = \sum_{\mathbf{p},r} p_\mu\left[\,-\,c_{\mathbf{p},r}\,u_{\mathbf{p},r}(x) + d^\dagger_{\mathbf{p},r}\,v_{\mathbf{p},r}(x)\right] \tag{4.67}$$

if and only if the canonical anticommutation relations (4.60) hold true.

From Noether theorem and canonical anticommutation relations (4.60) it follows that the classical vector current (4.53) is turned into the quantum operator

$$J^\mu(x) \;\equiv\; :q\,\bar\psi(x)\,\gamma^\mu\,\psi(x): \tag{4.68}$$

which satisfies the continuity operator equation

$$\partial_\mu J^\mu(x) = \frac{1}{\mathrm{i}\hbar}\,[\,P_\mu\,,\,J^\mu(x)\,] = 0 \tag{4.69}$$

As a consequence we have the conserved charge operator

$$Q \;\equiv\; \int \mathrm{d}\mathbf{x}\;\; :q\,\psi^\dagger(x)\,\psi(x): \;=\; q\sum_{\mathbf{p},r}\left(c^\dagger_{\mathbf{p},r}\,c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r}\,d_{\mathbf{p},r}\right) \tag{4.70}$$

whence it is manifest that the two types of quanta of the Dirac field do carry opposite charges. According to the customary convention for the Dirac spinor describing the electron positron field, we shall associate to particles the creation annihilation operators of the c-type and the negative electric charge $q = -\,e$ ($e > 0$), whilst the creation annihilation operators of the d-type and the positive electric charge $+e$ will be associated to the antiparticles, so that the electric charge operator becomes

$$Q_{\mathrm{e}} = \sum_{\mathbf{p},r}(-\,e)\left(c^\dagger_{\mathbf{p},r}\,c_{\mathbf{p},r} \;-\; d^\dagger_{\mathbf{p},r}\,d_{\mathbf{p},r}\right) \tag{4.71}$$

Finally, it turns out that also the helicity (4.49) of the Dirac field, which is a constant of motion, will be turned by the quantization procedure into the normal ordered operator expression

$$h \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\,z\;\;:\psi^\dagger(t,z)\,\Sigma_3\,\psi(t,z):$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p \sum_{r=1,\,2} \Big[c^{\dagger}_{p\,,r}\, c_{p\,,r}\, u^{\dagger}_{r}(p)\, \Sigma_3\, u_r(p) \\
&- \; d^{\dagger}_{p\,,r}\, d_{p\,,r}\, v^{\dagger}_{r}(p)\, \Sigma_3\, v_r(p) \Big]\, (2\omega_p)^{-1}
\end{aligned} \tag{4.72}
$$

where

$$
\{c_{p\,,r}\,,\, c^{\dagger}_{q\,,s}\} = \delta_{rs}\, \delta\,(p-q) = \{d_{p\,,r}\,,\, d^{\dagger}_{q\,,s}\}
$$
$$
(\,\forall\, r,s = 1,2 \quad p,q \in \mathbb{R}\,)
$$

and all other anticommutators vanish.

It is convenient to choose our standard spin-states, *i.e.* the orthogonal and normalized solutions of eq.s (4.15) and (4.16), in which we have to put $\mathbf{p} = (0,0,p)$. Then the helicity operator eventually becomes

$$
h \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p \,\Big[c^{\dagger}_{p\,,1}\, c_{p\,,1} - c^{\dagger}_{p\,,2}\, c_{p\,,2} + d^{\dagger}_{p\,,1}\, d_{p\,,1} - d^{\dagger}_{p\,,2}\, d_{p\,,2} \Big] \tag{4.73}
$$

The above expression does actually clarify the meaning of the polarization indices $r, s, \ldots = 1,2$. In conclusion, from the expressions (4.65), (4.66), (4.71) and (4.73), it follows that the operators $c^{\dagger}_{\mathbf{p}\,,r}$ and $c_{\mathbf{p}\,,r}$ do correspond respectively to the creation and annihilation operators for the particles of momentum $\mathbf{p}$, mass M with $p^2 = M^2$, electric charge $-\,e$, and positive helicity equal to $\frac{1}{2}$ $\;(r = 1)$ or negative helicity equal to $-\,\frac{1}{2}$ $\;(r = 2)$. Conversely, the operators $d^{\dagger}_{\mathbf{p}\,,r}$ and $d_{\mathbf{p}\,,r}$ will correspond respectively to the creation and annihilation operators for the antiparticles of momentum $\mathbf{p}$, mass M with $p^2 = M^2$, electric charge $+\,e$, and positive helicity equal to $\frac{1}{2}$ $\;(r = 1)$ or negative helicity equal to $-\,\frac{1}{2}$ $\;(r = 2)$.

The cyclic Fock vacuum is defined by

$$
c_{\mathbf{p}\,,r}\,|\,0\,\rangle \;=\; 0 \qquad d_{\mathbf{p}\,,r}\,|\,0\,\rangle \;=\; 0 \qquad (\,\forall\, r = 1,2 \quad \mathbf{p} \in \mathbb{R}^3\,) \tag{4.74}
$$

while the 1-particle energy momentum, helicity and charge eigenstates will correspond to

$$
|\,\mathbf{p}\; r\, -\,\rangle \equiv c^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle \qquad (\,r = 1,2 \quad \mathbf{p} \in \mathbb{R}^3\,) \tag{4.75}
$$

whereas the 1-antiparticle energy momentum, helicity and charge eigenstates will be

$$
|\,\mathbf{p}\; r\, +\,\rangle \equiv d^{\dagger}_{\mathbf{p}\,,r}\,|\,0\,\rangle \qquad (\,r = 1,2 \quad \mathbf{p} \in \mathbb{R}^3\,) \tag{4.76}
$$

Owing to the canonical anticommutation relations (4.60), it is impossible to accomodate two particles or two antiparticles in the very same quantum state as *e.g.*

$$
\begin{aligned}
c^{\dagger}_{\mathbf{p},r}\, c^{\dagger}_{\mathbf{p},r}\,|\,0\,\rangle &= -\,c^{\dagger}_{\mathbf{p},r}\, c^{\dagger}_{\mathbf{p},r}\,|\,0\,\rangle = 0 \\
d^{\dagger}_{\mathbf{p},r}\, d^{\dagger}_{\mathbf{p},r}\,|\,0\,\rangle &= -\,d^{\dagger}_{\mathbf{p},r}\, d^{\dagger}_{\mathbf{p},r}\,|\,0\,\rangle = 0
\end{aligned}
$$

As a consequence the multiparticle states do obey *Fermi-Dirac statistics* and the occupation numbers solely take the two possible values

$$
N_{\mathbf{p},r,\pm} = 0, 1 \qquad (\,r = 1, 2 \quad \mathbf{p} \in \mathbb{R}^3\,)
$$

which drives to the *Pauli exclusion principle* valid for all identical particles with half integer spin. The generic multiparticle state, that corresponds to an element of the basis of the Fock space, will be written in the form

$$
\prod_{a=1}^{A} \prod_{b=1}^{B} c^{\dagger}(\mathbf{p}_a\,,\, r_a\,)\, d^{\dagger}(\mathbf{p}_b\,,\, r_b\,)\,|\,0\,\rangle \tag{4.77}
$$

By virtue of (4.60) those states are completely antisymmetric with regard to the exchange of any pairs $(\mathbf{p}_a\,,\, r_a\,)$ and $(\mathbf{p}_b\,,\, r_b\,)$ and correspond to the presence of A particles and B antiparticles.

4.4 Covariance of the Quantized Spinor Field

Consider the general structure of the Poincaré transformations generators as normal ordered bilinear operator expressions, which can be readily obtained from the classical expressions (4.40), (4.41) and (4.42) : namely,

$$
\begin{aligned}
P_{\mu} &= \int \mathrm{d}\mathbf{x}\ \ :\psi^{\dagger}(x)\,\mathrm{i}\,\partial_{\mu}\,\psi(x): \\
M^{\,jk} &= \int \mathrm{d}\mathbf{x}\ \ :x^{j}\,\psi^{\dagger}(x)\,\mathrm{i}\,\partial^{k}\,\psi(x) - x^{k}\,\psi^{\dagger}(x)\,\mathrm{i}\,\partial^{j}\,\psi(x): \\
&+ \int \mathrm{d}\mathbf{x}\ \ :\psi^{\dagger}(x)\,\sigma^{\,jk}\,\psi(x): \\
M^{\,0k} &= x^{0}\,P^{k} - \int \mathrm{d}\mathbf{x}\; x^{k}\ :\psi^{\dagger}(x)\,\mathrm{i}\,\partial_{0}\,\psi(x):
\end{aligned} \tag{4.78}
$$

It can be verified by direct inspection that, owing to the anticommutation relations (4.60) and (4.64), those operator expressions indeed generate the infinitesimal Poincaré transformations for the quantized Dirac spinor field : namely,

$$\begin{aligned}\delta\psi(x) &= \mathrm{i}\,[\,P_\mu\,,\,\psi\,(x)\,]\,\epsilon^{\mu} - \tfrac{1}{2}\,\mathrm{i}\,[\,M_{\rho\sigma}\,,\,\psi\,(x)\,]\,\epsilon^{\,\rho\sigma}\\ &= \left[\,\epsilon^{\mu}\,\partial_\mu + \tfrac{1}{2}\,\epsilon_{\mu\nu}\,(\,x^{\nu}\,\partial^{\mu} - x^{\mu}\,\partial^{\nu}\,) + \tfrac{1}{2}\,\mathrm{i}\sigma^{jk}\,\epsilon_{jk}\,\right]\psi\,(x) \qquad (4.79)\end{aligned}$$

Moreover it is straightforward albeit quite heavy and tedious to verify that, by virtue of the canonical anticommutation relations, the operators (4.78) do indeed fulfill the Lie algebra (1.37) of the Poincaré group – for the explicit check see [10], **8.6**, pp. 163 - 165. The covariant 1-particle states and the corresponding creation annihilation operators can be defined in analogy with the construction (3.66) and take the form

$$|\,p\;r\,-\,\rangle = [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{\frac{1}{2}}\,c^{\dagger}_{\mathbf{p}\,,\,r}\,|\,0\,\rangle \equiv c^{\dagger}_{r}\,(p)\,|\,0\,\rangle \qquad (4.80)$$

$$\forall\;\mathbf{p}\in\mathbb{R} \qquad \forall\;r=1,2$$

$$|\,q\;s\,+\,\rangle = [\,(2\pi)^3\,2\omega_{\mathbf{q}}\,]^{\frac{1}{2}}\,d^{\dagger}_{\mathbf{q}\,,\,s}\,|\,0\,\rangle \equiv d^{\dagger}_{s}\,(p)\,|\,0\,\rangle \qquad (4.81)$$

$$\forall\;\mathbf{q}\in\mathbb{R} \qquad \forall\;s=1,2$$

which satisfy

$$\langle\,\pm\;s\;q\,|\,p\;r\;\pm\,\rangle \;=\; \delta_{rs}\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\delta\,(\mathbf{p}-\mathbf{q}) \qquad (4.82)$$

The normal mode decomposition of the Dirac field with respect to the new set of covariant creation annihilation operators becomes

$$\psi\,(x) = \sum_{r=1,2}\int Dp\,\left[\,c_r(p)\,u_r(p)\,\mathrm{e}^{-\,\mathrm{i}px} + d^{\dagger}_{r}(p)\,v_r(p)\,\mathrm{e}^{\mathrm{i}px}\,\right] \qquad (4.83)$$

where $p_\mu = (\omega_{\mathbf{p}}\,,\,-\,\mathbf{p})$ together with

$$(\not{p} - M)\,u_r(p) = 0 \qquad (\not{p} + M)\,v_r(p) = 0 \qquad r = 1,2 \qquad (4.84)$$

It is clear that the transformation law for the covariant 1-particle states and corresponding creation annihilation operators will be determined by the *unitary operators* associated to the Lorentz matrices. Actually, if we denote as usual the Lorentz matrices by $\Lambda(\omega) = \Lambda(\boldsymbol{\alpha},\boldsymbol{\eta})$ and by $U(\omega) = U(\boldsymbol{\alpha},\boldsymbol{\eta})$ the related unitary operators acting on the Fock space $\mathcal{F}$, where $(\boldsymbol{\alpha},\boldsymbol{\eta})$ are the canonical angular and rapidity coordinates of the restricted Lorentz group $L_{+}^{\uparrow} = O(1,3)_{+}^{+}$, then we can write

$$U(\omega)\,|\,p\;r\,\pm\,\rangle = |\,\Lambda\,p\;r\,\pm\,\rangle \qquad (4.85)$$

$$U(\omega)\,c_r\,(p)\,U^{\dagger}(\omega) = c_r\,(\Lambda\,p) \qquad (4.86)$$

$$
\begin{aligned}
U(\omega)\,c_r^\dagger(p)\,U^\dagger(\omega) &= c_r^\dagger(\Lambda\,p) && (4.87)\\
U(\omega)\,d_r(p)\,U^\dagger(\omega) &= d_r(\Lambda\,p) && (4.88)\\
U(\omega)\,d_r^\dagger(p)\,U^\dagger(\omega) &= d_r^\dagger(\Lambda\,p) && (4.89)\\
\forall\ \mathbf{p}\in\mathbb{R} \qquad \forall\ r = 1,2
\end{aligned}
$$

so that

$$
\begin{aligned}
\langle\pm\ s\ \Lambda\,q\,|\,\Lambda\,p\ r\ \pm\rangle &= \\
\delta_{rs}\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\delta(\mathbf{p}-\mathbf{q}) &= \langle\pm\ s\ q\,|\,p\ r\ \pm\rangle && (4.90)
\end{aligned}
$$

As a consequence we can eventually write

$$
\begin{aligned}
\psi'(x) &\equiv U(\omega)\,\psi(x)\,U^\dagger(\omega)\\
&= \sum_{r=1,2}\int Dp\,\Big[U(\omega)\,c_r(p)\,U^\dagger(\omega)\,u_r(p)\,\mathrm{e}^{-\mathrm{i}px}\\
&+ U(\omega)\,d_r^\dagger(p)\,U^\dagger(\omega)\,v_r(p)\,\mathrm{e}^{\mathrm{i}px}\Big]\\
&= \sum_{r=1,2}\int Dp\,\Big[c_r(\Lambda p)\,u_r(p)\,\mathrm{e}^{-\mathrm{i}px} + d_r^\dagger(\Lambda p)\,v_r(p)\,\mathrm{e}^{\mathrm{i}px}\Big]\\
&= \sum_{r=1,2}\int Dp'\,\Big[c_r(p')\,u_r(\Lambda^{-1}p')\,\mathrm{e}^{-\mathrm{i}p'x'}\\
&+ d_r^\dagger(p')\,v_r(\Lambda^{-1}p')\,\mathrm{e}^{\mathrm{i}p'x'}\Big]\\
&= \sum_{r=1,2}\int Dp'\,\Big[c_r(p')\,\Lambda_{\frac12}^{-1}\,u_r(p')\,\mathrm{e}^{-\mathrm{i}p'x'}\\
&+ d_r^\dagger(p')\,\Lambda_{\frac12}^{-1}\,v_r(p')\,\mathrm{e}^{\mathrm{i}p'x'}\Big] = \Lambda_{\frac12}^{-1}\,\psi(x') && (4.91)
\end{aligned}
$$

where the transformation law (4.23) of the spin-states has been used. In addition, taking into account formulæ (4.67) and (4.78), from the canonical anticommutation relations (4.64) we eventually obtain for a quite general Poincaré transformation

$$
\begin{aligned}
\psi'(x) &= U(a,\omega)\,\psi(x)\,U^\dagger(a,\omega)\\
&= \Lambda_{\frac12}^{-1}(\omega)\,\psi(\Lambda\,x+a) && (4.92)\\
U(a,\omega) &\equiv \exp\left\{\mathrm{i}\,a^\mu P_\mu - \tfrac12\,\mathrm{i}\,\omega^{\rho\sigma}M_{\rho\sigma}\right\} && (4.93)
\end{aligned}
$$

which shows that the general transformation law of the quantized spinor field consists in an infinite dimensional irreducible **unitary** representation of the

Poincaré group acting on the Fock space $\mathcal{F}$. In particular, for an infinitesimal Poincaré transformation we find

$$\begin{aligned}\psi(x)+\delta\psi(x) &= \psi(x)+\mathrm{i}\,[\,P_\mu\,,\,\psi(x)\,]\,\epsilon^\mu-\tfrac{1}{2}\,\mathrm{i}\,[\,M_{\rho\sigma}\,,\,\psi(x)\,]\,\epsilon^{\,\rho\sigma}\\ &= \psi(x)+\epsilon^\mu\,\partial_\mu\psi(x)+\epsilon^{\,\rho\sigma}\,x_\sigma\,\partial_\rho\psi(x)\\ &+\ \tfrac{1}{2}\,\mathrm{i}\,\epsilon_{\,jk}\,\sigma^{\,jk}\,\psi(x)\end{aligned}$$

where $\delta\ a^\mu\ =\ \epsilon^\mu$ and $\delta\ \omega^{\,\rho\sigma}\ =\ \epsilon^{\,\rho\sigma}$ are the infinitesimal parameters. In particular, from the finite transformation rule (2.78), it is simple to check, for instance, that the mass operator is Lorentz invariant, *i.e.*

$$\bar{\psi}'(x)\,\psi'(x)=\bar{\psi}(x')\,\psi(x')$$

The reader should gather the difference between the transformation law (4.92) for the quantized Dirac spinor field and the transformation law (2.75) of the classical relativistic Dirac spinor wave field under restricted Lorentz group, *i.e.*

$$\psi'(x')\ =\ \Lambda_{\frac{1}{2}}(\omega)\,\psi(x)\ =\ \exp\left\{-\tfrac{1}{2}\,\mathrm{i}\,\sigma^{\mu\nu}\,\omega_{\,\mu\nu}\right\}\,\psi(x)\qquad x'=\Lambda\,x$$

which is unitary with respect to the inner product

$$(\,\psi_2\,,\,\psi_1\,)=\int\mathrm{d}\mathbf{x}\,\psi_2^\dagger(t,\mathbf{x})\,\psi_1(t,\mathbf{x})\tag{4.94}$$

Notice that the density

$$\varrho(x)=\psi^\dagger(t,\mathbf{x})\,\psi(t,\mathbf{x})$$

does represent the positive semidefinite probability density in the old Dirac theory, *i.e.* the relativistic quantum mechanics of the electron. Conversely, the corresponding operator valued local density

$$\widehat{\varrho}(x)=q:\psi^\dagger(t,\mathbf{x})\,\psi(t,\mathbf{x}):$$

has not a definite sign for it represents the charge density of the quantized spinor field, see equation (4.70).

Beside the transformation laws of the quantized Dirac field under the continuous Poincarè group, a very important role in the Standard Model of the fundamental interactions for Particle Physics is played by the charge conjugation, parity and time reversal discrete transformations, the so called CPT symmetries, on the quantized spinor matter fields. We shall analyze in details the latter ones at the end of the present chapter.

4.5 Special Distributions

From the canonical anticommutation relations (4.60) and the normal mode expansion (4.28) of the Dirac field, the so called *canonical anticommutator at arbitrary points* between two free Dirac spinor field operators can be readily shown to be equal to zero

$$\{\psi_\alpha(x)\,,\,\psi_\beta(y)\} = 0 = \{\bar{\psi}_\alpha(x)\,,\,\bar{\psi}_\beta(y)\} \qquad (\,\alpha,\beta = 1L, 2L, 1R, 2R\,)$$

On the contrary, the canonical anticommutator at arbitrary points between the free Dirac field and its adjoint does not vanish: it can be easily calculated to be

$$\{\psi_\alpha(x)\,,\,\bar{\psi}_\beta(y)\} \equiv S_{\alpha\beta}(x-y) = -\,\mathrm{i}\,(\,\mathrm{i}\,\partial\!\!\!/_x + M\,)_{\alpha\beta}\,D(x-y) \tag{4.95}$$

where $D(x-y)$ is the Pauli-Jordan distribution of the real scalar field.

`Proof.` From the normal modes expansions (4.62) of the spinor fields

$$\begin{aligned}
\psi_\alpha(x) &= \sum_{\mathbf{p}\,,\,r}\left[\,(2\pi)^3 2\omega_{\mathbf{p}}\right]^{-\frac{1}{2}} \\
&\times \left[c_{\mathbf{p}\,,\,r}\,u_{\alpha\,,\,r}(\mathbf{p})\,\mathrm{e}^{-\,\mathrm{i}\,p\,x} + d^{\,\dagger}_{\mathbf{p}\,,\,r}\,v_{\alpha\,,\,r}(\mathbf{p})\,\mathrm{e}^{\,\mathrm{i}\,p\,x}\right]_{p_0\,=\,\omega_{\mathbf{p}}} \\
\bar{\psi}_\beta(y) &= \sum_{\mathbf{q}\,,\,s}\left[\,(2\pi)^3 2\omega_{\mathbf{q}}\right]^{-\frac{1}{2}} \\
&\times \left[c^{\,\dagger}_{\mathbf{q}\,,\,s}\,\bar{u}_{\beta\,,\,s}(\mathbf{q})\,\mathrm{e}^{\,\mathrm{i}\,q\,y} + d_{\mathbf{q}\,,\,s}\,\bar{v}_{\beta\,,\,s}(\mathbf{q})\,\mathrm{e}^{-\,\mathrm{i}\,q\,y}\right]_{q_0\,=\,\omega_{\mathbf{q}}}
\end{aligned}$$

and the canonical anticommutation relations (4.60) one finds

$$\begin{aligned}
\{\psi_\alpha(x)\,,\,\bar{\psi}_\beta(y)\} &= \int \mathrm{D}p \sum_{r\,=\,1,2}\Big[\,u_{\alpha\,,\,r}(\mathbf{p})\,\bar{u}_{\beta\,,\,r}(\mathbf{p})\,\mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)} \\
&+\; v_{\alpha\,,\,r}(\mathbf{p})\,\bar{v}_{\beta\,,\,r}(\mathbf{p})\,\mathrm{e}^{\,\mathrm{i}\,p\,(x-y)}\Big]_{p_0\,=\,\omega_{\mathbf{p}}}
\end{aligned}$$

Taking into account the sums over the spin states (4.35)

$$\sum_{r\,=\,1,2}\left\{\begin{array}{l} u_{\alpha\,,\,r}(\mathbf{p})\,\bar{u}_{\beta\,,\,r}(\mathbf{p}) = (\,p\!\!\!/ + M\,)_{\alpha\beta} \\ v_{\alpha\,,\,r}(\mathbf{p})\,\bar{v}_{\beta\,,\,r}(\mathbf{p}) = (\,p\!\!\!/ - M\,)_{\alpha\beta}\end{array}\right. \qquad (\,p_0 = \omega_{\mathbf{p}}\,)$$

we readily come to the expression, by omitting the spinorial indices for the sake of brevity,

$$\begin{aligned}
\{\psi(x)\,,\,\bar{\psi}(y)\} &\equiv S(x-y) \\
&= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\left(\frac{p\!\!\!/ + M}{2\,\omega_{\mathbf{p}}}\,\mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)} - \mathrm{e}^{\,\mathrm{i}\,p\,(x-y)}\,\frac{M - p\!\!\!/}{2\,\omega_{\mathbf{p}}}\right)_{p_0\,=\,\omega_{\mathbf{p}}} \\
&= (\mathrm{i}\,\partial\!\!\!/_x + M)\int \mathrm{D}p\,\left(\mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)} - \mathrm{e}^{\,\mathrm{i}\,p\,(x-y)}\right)_{p_0\,=\,\omega_{\mathbf{p}}} \\
&= (\mathrm{i}\,\partial\!\!\!/_x + M)\int \frac{\mathrm{d}^4 p}{(2\pi)^3}\,\mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)}\,\delta(p^2 - M^2)\,\mathrm{sgn}(p_0) \\
&= -\,\mathrm{i}\,(\mathrm{i}\,\partial\!\!\!/_x + M)\,D(x-y)
\end{aligned} \tag{4.96}$$

where use has been made of the formulæ (4.34). *Quod Erat Demonstrandum*

The canonical anticommutator at arbitrary points (4.96) is a solution of the Dirac equation which does not vanish when $(x-y)$ is a spacelike interval with $x^0 \neq y^0$. In fact, at variance with the real scalar field case in which

$$D\,(x-y) \equiv 0 \qquad \forall\,(x_0-y_0)^2 < (\mathbf{x}-\mathbf{y})^2 \qquad (x^0 \neq y^0) \tag{4.97}$$

we find instead the nonvanishing equal time anticommutator

$$S\,(0, \mathbf{x}-\mathbf{y}) = \gamma^0\,\delta\,(\mathbf{x}-\mathbf{y}) \tag{4.98}$$

in agreement with (4.64).

The causal Green's function or Feynman propagator for the Dirac field is

$$\begin{aligned} \langle\,0\,|\,T\,\psi_\alpha\,(x)\,\bar\psi_\beta\,(y)\,|\,0\,\rangle &= \begin{cases} \langle\,0\,|\,\psi_\alpha\,(x)\,\bar\psi_\beta\,(y)\,|\,0\,\rangle & \text{for } x^0 > y^0 \\ -\,\langle\,0\,|\,\bar\psi_\beta\,(y)\,\psi_\alpha\,(x)\,|\,0\,\rangle & \text{for } x^0 < y^0 \end{cases} \\ &\equiv \mathrm{i}\,S^c_{\alpha\beta}\,(x-y) = S^F_{\alpha\beta}\,(x-y) \end{aligned}$$

Actually we have

$$S^F\,(x-y) = (\mathrm{i}\,\partial\!\!\!/_x + M)\;D_F\,(x-y) \tag{4.99}$$

where $D_F\,(x-y)$ is the Feynman propagator of the real scalar field.

`Proof.` From the very definition as the vacuum expectation value of the chronological product of spinor fields we can write

$$\begin{aligned} S^F\,(x-y) &= \theta\,(x^0-y^0)\,\langle\,\psi\,(x)\,\bar\psi\,(y)\,\rangle_0 - \theta\,(y^0-x^0)\,\langle\,\bar\psi\,(y)\,\psi\,(x)\,\rangle_0 \\ &= \theta\,(x^0-y^0)\sum_{\mathbf{p}\,,\,r}[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,u_r\,(\mathbf{p})\otimes\bar u_r\,(\mathbf{p}) \\ &\times\ \exp\{-\,\mathrm{i}\omega_{\mathbf{p}}\,(x^0-y^0) + \mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \\ &-\ \theta\,(y^0-x^0)\sum_{\mathbf{p}\,,\,r}[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-1}\,v_r\,(\mathbf{p})\otimes\bar v_r\,(\mathbf{p}) \\ &\times\ \exp\{+\,\mathrm{i}\omega_{\mathbf{p}}\,(x^0-y^0) - \mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\} \end{aligned} \tag{4.100}$$

Hence, from the sums over the spin-states (4.34) we obtain

$$\begin{aligned} S^F\,(x-y) &= \theta\,(x^0-y^0)\int Dp\ \Big[\,(M+p\!\!\!/)\,\mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)}\,\Big]_{p_0\,=\,\omega_{\mathbf{p}}} \\ &+\ \theta\,(y^0-x^0)\int Dp\ \Big[\,(M-p\!\!\!/)\,\mathrm{e}^{\mathrm{i}\,p\,(x-y)}\,\Big]_{p_0\,=\,\omega_{\mathbf{p}}} \\ &= \theta\,(x^0-y^0)\ (\mathrm{i}\,\partial\!\!\!/_x + M)\int Dp\ \ \mathrm{e}^{-\,\mathrm{i}\,p\,(x-y)}\,\Big]_{p_0\,=\,\omega_{\mathbf{p}}} \\ &+\ \theta\,(y^0-x^0)\ (\mathrm{i}\,\partial\!\!\!/_x + M)\int Dp\ \ \mathrm{e}^{\mathrm{i}\,p\,(x-y)}\,\Big]_{p_0\,=\,\omega_{\mathbf{p}}} \end{aligned}$$

and if we recall the definitions

$$\begin{aligned}
\pm\,\mathrm{i}\,D^{(\pm)}(x-y) &= \int Dp\ \ \mathrm{e}^{\pm\,\mathrm{i}\,p\,(x-y)}\Big\rfloor_{p_0\,=\,\omega_{\mathbf{p}}} \\
&= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3\,2\omega_{\mathbf{p}}}\ \exp\{\pm\mathrm{i}\omega_{\mathbf{p}}\,(x^0-y^0)\mp\mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\}
\end{aligned}$$

we readily come to the expessions

$$\begin{aligned}
& S^F\,(x-y) = \theta\,(x^0-y^0)\ (\mathrm{i}\,\partial\!\!\!/_x + M)\ \frac{1}{\mathrm{i}}\,D^{(-)}\,(x-y) \\
+\ \ & \mathrm{i}\,\theta\,(y^0-x^0)\ (\mathrm{i}\,\partial\!\!\!/_x + M)\ D^{(+)}\,(x-y) \\
=\ \ & (\mathrm{i}\,\partial\!\!\!/_x + M)\ \mathrm{i}\,\theta\,\big(y^0-x^0\big)\ D^{(+)}\,(x-y) - (\mathrm{i}\,\partial\!\!\!/_x + M)\ \mathrm{i}\,\theta\,\big(x^0-y^0\big)\ D^{(-)}\,(x-y) \\
-\ \ & \gamma^0\,\delta\big(x^0-y^0\big)\ \Big[D^{(-)}\,(x-y) + D^{(+)}\,(x-y)\Big]\ =\ (\mathrm{i}\,\partial\!\!\!/_x + M)\ D_F\,(x-y)
\end{aligned}$$

where I did make use of the Pauli-Jordan distribution property

$$D\,(x)\ =\ D^{(+)}\,(x) + D^{(-)}\,(x)\,, \qquad\qquad D\,(0,\mathbf{x}) = 0$$

and of the relation (3.89)

$$D_F(x-y) = \mathrm{i}\,\theta\,\big(y^0-x^0\big)\ D^{(+)}\,(x-y) - \mathrm{i}\,\theta\,\big(x^0-y^0\big)\ D^{(-)}\,(x-y)$$

Quod Erat Demonstrandum

The Fourier representation of the spinorial Feynman propagator reads

$$\begin{aligned}
S^F\,(x-y) &= (\mathrm{i}\,\partial\!\!\!/_x + M)\,D_F\,(x-y) \\
&= \frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}p\ \frac{p\!\!\!/ + M}{p^2-M^2+\mathrm{i}\varepsilon}\ \exp\{-\mathrm{i}p\cdot(x-y)\} \\
&= \frac{1}{(2\pi)^4}\int \mathrm{d}p\ \frac{\mathrm{i}}{p\!\!\!/ - M + \mathrm{i}\varepsilon}\ \exp\{-\mathrm{i}p\cdot(x-y)\} \qquad (4.101)
\end{aligned}$$

and consequently

$$(\mathrm{i}\,\partial\!\!\!/_x - M)\,S^F_{\alpha\beta}\,(x-y)\ =\ \mathrm{i}\,\delta(x-y) \qquad (4.102)$$

$$\frac{\mathrm{i}\,(\,p\!\!\!/ + M\,)_{\alpha\beta}}{p^2-M^2+\mathrm{i}\varepsilon}\ \stackrel{\mathrm{def}}{=}\ \left(\frac{\mathrm{i}}{p\!\!\!/ - M}\right)_{\alpha\beta} \qquad (4.103)$$

It is also convenient to write the adjoint form of the inhomogeneous equation for the Feynman propagator of the spinor field. To this concern, let us first obtain the hermitean conjugate of equation (4.102) *viz.*,

$$\begin{aligned}
\mathrm{i}\,\delta\,(x-y) &= \mathrm{i}\,(\partial/\partial x^\mu)\,S_F^\dagger(x-y)\,\gamma^{\mu\,\dagger} + M\,S_F^\dagger(x-y) \\
&= \mathrm{i}\,(\partial/\partial x^\mu)\,S_F^\dagger(x-y)\,\gamma^0\,\gamma^\mu\,\gamma^0 + M\,S_F^\dagger(x-y) \quad (4.104)
\end{aligned}$$

Multiplication by γ^0 from left and right yields

$$\begin{aligned} \mathrm{i}\,\delta\,(x-y) &= \mathrm{i}\,\gamma^0\,(\partial/\partial x^\mu)\,S_F^\dagger(x-y)\,\gamma^0\,\gamma^\mu + \gamma^0\,M\,S_F^\dagger(x-y)\,\gamma^0 \\ &\stackrel{\mathrm{def}}{=} \bar{S}^{\,F}(\,y-x)\,(\,\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_x + M\,) \end{aligned} \tag{4.105}$$

where

$$\begin{aligned} \bar{S}^{\,F}(\,y-x) &= \gamma^0\,S_F^\dagger(x-y)\,\gamma^0 \\ &= \frac{-\,\mathrm{i}}{(2\pi)^4}\int \mathrm{d}p\;\frac{p\!\!\!/ + M}{p^2 - M^2 - \mathrm{i}\varepsilon}\;\exp\{-\,\mathrm{i}\,p\cdot(y-x)\} \end{aligned} \tag{4.106}$$

4.5.1 The Euclidean Fermions

Also for the present fermion propagator we can safely perform the Wick rotation $i\,p^0 = p_4\,,\ ix^0 = x_4$ that yields

$$\begin{aligned} S^{\,F}_{\alpha\beta}(\,-\mathrm{i}x_4,\mathbf{x}) &= \mathrm{i}\,(2\pi)^{-4}\int \mathrm{d}\mathbf{p}\int_{-\infty}^{\infty}\mathrm{d}p_4 \\ \frac{\exp\{\mathrm{i}p_4\,x_4 + \mathrm{i}\,\mathbf{p}\cdot\mathbf{x}\}}{p_4^2 + \mathbf{p}^2 + M^2}&\,(\,\gamma^0\,p_4 - \mathrm{i}\gamma^k\,p^k + \mathrm{i}M\,)_{\alpha\beta} \end{aligned}$$

It follows that if we define the *hermitean euclidean gamma matrices*

$$\bar{\gamma}_\mu = \left(\bar{\gamma}_k\,,\,\bar{\gamma}_4\right) \qquad \bar{\gamma}_4 \equiv\ \gamma^0 \qquad \bar{\gamma}_k \equiv\ -\,\mathrm{i}\gamma^k \quad (\,k=1,2,3\,) \tag{4.107}$$

$$\bar{\gamma}_\mu = \bar{\gamma}_\mu^{\,\dagger} \qquad \left\{\bar{\gamma}_\mu\,,\,\bar{\gamma}_\nu\right\} = 2\,\delta_{\mu\nu} \tag{4.108}$$

or even more explicitly

$$\bar{\gamma}_1 = \begin{pmatrix} 0 & -i\,\sigma_1 \\ i\,\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{4.109}$$

$$\bar{\gamma}_2 = \begin{pmatrix} 0 & -i\,\sigma_2 \\ i\,\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.110}$$

$$\bar{\gamma}_3 = \begin{pmatrix} 0 & -i\,\sigma_3 \\ i\,\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{4.111}$$

$$\bar{\gamma}_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.112}$$

Then we can write

$$\begin{aligned} S^F_{\alpha\beta}(-\mathrm{i}x_4, \mathbf{x}) &= \frac{\mathrm{i}}{(2\pi)^4} \int \mathrm{d}p_E \; \frac{\exp\{\mathrm{i}p_E \cdot x_E\}}{p_E^2 + M^2} \; (\bar{\gamma}_\mu \; p_{E\mu} + \mathrm{i}M)_{\alpha\beta} \\ &= \frac{\mathrm{i}}{(2\pi)^4} \int \mathrm{d}p_E \; \exp\{\mathrm{i}p_E \cdot x_E\} \; \left(\frac{1}{\not{p}_E - \mathrm{i}M} \right)_{\alpha\beta} \\ &= (\not{\partial}_E - M)_{\alpha\beta} \; D_E(x_E) \\ &\equiv -S^E_{\alpha\beta}(x_E) \end{aligned} \tag{4.113}$$

where we understand

$$\bar{\gamma}_\mu p_{E\mu} \equiv \not{p}_E \qquad \mathrm{i}\not{\partial}_E \equiv \bar{\gamma}_\mu \; \frac{\mathrm{i}\partial}{\partial x_{E\mu}}$$

This suggests that the proper classical variables for the setting up of an euclidean formulation of the Dirac spinor field theory, in analogy with what we have already seen in the scalar field case, should be two euclidean bispinors ψ_E and $\bar{\psi}_E$ obeying

$$\{\psi_E(x)\,,\,\psi_E(y)\} = \{\bar{\psi}_E(x)\,,\,\bar{\psi}_E(y)\} = \{\psi_E(x)\,,\,\bar{\psi}_E(y)\} = 0 \tag{4.114}$$

for all points x and y of the four dimensional euclidean space $\mathbb{R}^4$.

The last of these relations is crucial, for it implies that $\bar{\psi}_E$ does not necessarily coincide with the adjoint of ψ times some matrix γ_4. Thus, if we want to set up a meaningful euclidean formulation for the Dirac spinor field theory, then we *can treat ψ_E and $\bar{\psi}_E$ as totally independent classical Grassmann valued variables.* This independence is the main novelty of the euclidean fermion field theory; the rest of the construction is straightforward.

For instance, we use the definition of the hermitean euclidean gamma matrices to derive the $O(4)$ transformation law for ψ_E in the usual way[3], while define $\bar{\psi}_E$ to transform like the transposed of ψ_E. Next, we define $\bar{\gamma}_5$, a hermitean matrix, by

$$\bar{\gamma}_5 = \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_4 = \bar{\gamma}_5^\dagger = -\gamma_5$$

[3]Remember that the orthogonal group $O(4)$ of the rotations in the euclidean space $\mathbb{R}^4$ is a semisimple Lie group $O(4) = O(3)_L \times O(3)_R$.

Thus, $\bar\psi_E \psi_E$ is a scalar, $\bar\psi_E\,\bar\gamma_5\,\psi_E$ a pseudoscalar, $\bar\psi_E\,\bar\gamma_\mu\,\psi_E$ a vector *etc.*

The euclidean action for the free Dirac field is given by

$$S_E[\,\psi_E,\bar\psi_E\,] \;=\; \int \mathrm{d}^4x_E\;\bar\psi_E(x_E)\,(\,\partial\!\!\!/_E + M\,)\;\psi_E(x_E) \tag{4.115}$$

Here the overall sign is purely conventional : we could always absorb it into ψ_E if we wanted to (remember that we are free to change ψ_E without touching $\bar\psi_E$). Conversely, the lack of the factor i in front of the derivative term is not at all conventional : it is there just to ensure that the euclidean fermion propagator, which is named 2-point *Schwinger's function*, is proportional to $(\mathrm{i}p\!\!\!/_E - M)/(p_E^2 + M^2)$; if it were not for this i , then we would have tachyon poles after a Wick rotation back to the Minkowski momentum space.

It is worthwhile to notice that the above Dirac euclidean action can be obtained from the corresponding one in the Minkowski spacetime, after the customary standard replacements

$$x_4 = \mathrm{i}\,x_0 \qquad \bar\gamma_k \equiv\; -\,\mathrm{i}\gamma^k \quad (\,k=1,2,3\,) \qquad \bar\gamma_4 \equiv\; \gamma^0$$
$$\psi(x) \;\mapsto\; \psi_E(x_E) \qquad \bar\psi(x) \;\mapsto\; \bar\psi_E(x_E)$$

As a matter of fact we readily obtain

$$\begin{aligned} S[\,\psi,\bar\psi\,] \;&\mapsto\; \mathrm{i}\,S_E[\,\psi_E,\bar\psi_E\,] \\ &=\; \int \mathrm{d}^4x_E\;\bar\psi_E(x_E)\,(\,\mathrm{i}\partial\!\!\!/_E + \mathrm{i}M\,)\;\psi_E(x_E) \end{aligned} \tag{4.116}$$

Furthermore, the euclidean Dirac operator $(\,\partial\!\!\!/_E + M\,)$ is precisely that one which gives, according to the definition (4.113), the 2-point Schwinger function inversion formula[4]

$$(\,\partial\!\!\!/_E + M\,)_{\alpha\beta}\;S^E_{\beta\eta}(x_E) \;=\; \delta(x_E)\;\delta_{\alpha\eta}$$

Finally, it is worthwhile to remark that the euclidean action for the free Dirac field is not a real quantity. As we shall see, this fact will not cause any troubles in the analytic continuation to the Minkowski spacetime.

[4]The $n-$point Green functions in the euclidean space are usually named the $n-$point Schwinger functions.

To end up we have

$$
\begin{aligned}
S_E^0[\,\phi_E\,] \;&=\; \int \mathrm{d}x_E \;\left[\,\tfrac{1}{2}\,\partial_\mu\,\phi_E\,\partial_\mu\,\phi_E + \tfrac{1}{2}\,m^2\,\phi_E^2\,\right] \\
&\doteq\; \int \mathrm{d}x_E\;\tfrac{1}{2}\,\phi_E\,(x_E)\left(-\,\partial^2 + m^2\right)\phi_E\,(x_E) \qquad (4.117) \\
S_E^0[\,\bar{\psi}\,,\,\psi_E\,] \;&=\; \int \mathrm{d}x_E\;\bar{\psi}\,(x_E)\,(\,\not{\partial}_E + M\,)\,\psi_E\,(x_E) \qquad (4.118)
\end{aligned}
$$

in such a manner that we can summarize the useful relationships

$$D_F\,(-\,\mathrm{i}x_4\,,\,\mathbf{x}) \;\to\; -\,D_E\,(x_E) \qquad (4.119)$$

$$D_E\,(x_E) \;=\; \frac{1}{(2\pi)^4}\int \mathrm{d}\,k_E\;\frac{\exp\{\,\mathrm{i}k_E\cdot x_E\,\}}{k_E^2+m^2} \qquad (4.120)$$

$$\left(-\,\partial_E^2 + m^2\right)\,D_E\,(x_E) \;=\; \delta\,(x_E) \qquad (4.121)$$

$$S^F_{\alpha\beta}\,(-\mathrm{i}x_4,\mathbf{x}) \;\to\; -\,S^E_{\alpha\beta}\,(x_E) \qquad (4.122)$$

$$S^E_{\alpha\beta}\,(x_E) \;=\; \int \frac{\mathrm{d}\,p_E}{(2\pi)^4}\;\exp\{\,\mathrm{i}p_E\cdot x_E\,\}\;\left(\frac{\mathrm{i}}{-\,\not{p}_E + \mathrm{i}M}\right)_{\alpha\beta} \qquad (4.123)$$

$$(\,\not{\partial}_E + M\,)_{\alpha\beta}\;S^E_{\beta\eta}\,(x_E) \;=\; \delta\,(x_E)\,\delta_{\alpha\eta} \qquad (4.124)$$

4.6 The Fermion Generating Functional

4.6.1 Symanzik Equations for Fermions

Consider the vacuum expectation value

$$Z_0[\,\zeta\,,\,\bar\zeta\,] \equiv \left\langle T\,\exp\left\{ \mathrm{i}\int \mathrm{d}\,x\,\left[\,\bar\zeta(x)\,\psi\,(x) + \bar\psi\,(x)\,\zeta(x)\,\right]\right\}\right\rangle_0 \tag{4.125}$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{i}^{\,n}}{n!}\int \mathrm{d}\,x_1 \int \mathrm{d}\,x_2\,\cdots \int \mathrm{d}\,x_n \int \mathrm{d}\,y_1 \int \mathrm{d}\,y_2\,\cdots \int \mathrm{d}\,y_n$$
$$\left\langle T\,\Big(\bar\zeta(x_1)\,\psi\,(x_1) + \bar\psi\,(y_1)\,\zeta(y_1)\Big)\,\cdots\,\Big(\bar\zeta(x_n)\,\psi\,(x_n) + \bar\psi\,(y_n)\,\zeta(y_n)\Big)\right\rangle_0$$

the suffix zero denoting the free field theory, where $\zeta(x)$ and $\bar\zeta(x)$ are the so called classical fermion external sources, which turn out to be *Grassmann valued functions* with the canonical dimensions $[\,\zeta\,] = \mathrm{cm}^{\,-5/2}$ and which satisfy

$$\{\zeta(x)\,,\,\zeta(y)\} = \{\bar\zeta(x)\,,\,\bar\zeta(y)\} = 0 \tag{4.126}$$
$$\{\zeta(x)\,,\,\bar\zeta(y)\} = \{\bar\zeta(x)\,,\,\zeta(y)\} = 0 \tag{4.127}$$
$$\{\zeta(x)\,,\,\psi\,(y)\} = \{\bar\zeta(x)\,,\,\bar\psi\,(y)\} = 0 \tag{4.128}$$
$$\{\zeta(x)\,,\,\bar\psi\,(y)\} = \{\bar\zeta(x)\,,\,\psi\,(y)\} = 0 \tag{4.129}$$

Notice that, just owing to (4.129), the products $\bar\zeta(x)\,\psi\,(x)$ and $\bar\psi\,(y)\,\zeta(y)$ do not take up signs under time ordering, *i.e.*

$$T\Big(\bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y)\Big) = \begin{cases} \bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y) & \text{if } x^0 > y^0 \\ \bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x) & \text{if } x^0 < y^0 \end{cases}$$

In fact

$$\bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y) = \bar\zeta(x)\,\zeta(y)\,\psi\,(x)\,\bar\psi\,(y) \qquad (x^0 > y^0)$$
$$\bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x) = \bar\zeta(x)\,\zeta(y)\,\big(-\,\bar\psi\,(y)\,\psi\,(x)\big) \qquad (x^0 < y^0)$$

so that

$$\begin{aligned} \left\langle T\Big(\bar\zeta(x)\,\psi\,(x)\,\bar\psi\,(y)\,\zeta(y)\Big)\right\rangle_0 &= \bar\zeta(x)\,\zeta(y)\,\left\langle T\,\psi\,(x)\,\bar\psi\,(y)\right\rangle_0 \\ &= \bar\zeta(x)\,\zeta(y)\,S_F(x-y) \\ &= \left\langle T\Big(\bar\psi\,(y)\,\zeta(y)\,\bar\zeta(x)\,\psi\,(x)\Big)\right\rangle_0 \end{aligned}$$

The functional differentiation with respect to the classical Grassmann valued sources is defined by

$$
\begin{aligned}
\{\delta/\delta\bar\zeta(x)\,,\,\bar\zeta(y)\} &= \delta(x-y) = \{\delta/\delta\zeta(x)\,,\,\zeta(y)\}\\
\{\delta/\delta\bar\zeta(x)\,,\,\zeta(y)\} &= 0 = \{\delta/\delta\zeta(x)\,,\,\bar\zeta(y)\}\\
\{\delta/\delta\bar\zeta(x)\,,\,\delta/\delta\bar\zeta(y)\} &= 0 = \{\delta/\delta\zeta(x)\,,\,\delta/\delta\zeta(y)\}\\
\{\delta/\delta\bar\zeta(x)\,,\,\delta/\delta\zeta(y)\} &= 0 = \{\delta/\delta\zeta(x)\,,\,\delta/\delta\bar\zeta(y)\}
\end{aligned}
\tag{4.130}
$$

where all operators act on their right. It follows that

$$
\begin{aligned}
&-\,\mathrm{i}\,\delta Z_0[\,\zeta\,,\,\bar\zeta\,]/\delta\bar\zeta(x) \;=\\
&\left\langle T\,\psi\,(x)\exp\left\{i\int \mathrm{d}y\,\left[\,\bar\zeta(y)\,\psi\,(y)+\bar\psi\,(y)\,\zeta(y)\,\right]\right\}\right\rangle_0\\
&\mathrm{i}\,\delta Z_0[\,\zeta\,,\,\bar\zeta\,]/\delta\zeta(x) \;=\\
&\left\langle T\,\bar\psi\,(x)\exp\left\{\mathrm{i}\int \mathrm{d}y\,\left[\,\bar\zeta(y)\,\psi\,(y)+\bar\psi\,(y)\,\zeta(y)\,\right]\right\}\right\rangle_0
\end{aligned}
\tag{4.131}
$$

where the plus sign in the second equality is because

$$
\begin{aligned}
\delta/\delta\zeta(x)\int \mathrm{d}y\,\bar\psi\,(y)\,\zeta(y) &= \int \mathrm{d}y\,\delta/\delta\zeta(x)[\,\bar\psi\,(y)\,\zeta(y)\,]\\
&= \int \mathrm{d}y\,\bar\psi\,(y)[\,-\,\delta\zeta(y)/\delta\zeta(x)\,]\\
&= -\int \mathrm{d}y\,\bar\psi\,(y)\,\delta(x-y)\\
&= -\,\bar\psi\,(x)
\end{aligned}
\tag{4.132}
$$

Taking one more functional derivatives of the generating functional (4.125) we find

$$
\begin{aligned}
&(\delta/\delta\bar\zeta(x))\,\delta Z_0[\,\zeta\,,\,\bar\zeta\,]\,/\,\delta\zeta\,(y) \;= \qquad\qquad (4.133)\\
&\left\langle T\,\psi\,(x)\,\bar\psi\,(y)\,\exp\left\{\mathrm{i}\int \mathrm{d}y\,\left[\,\bar\zeta(y)\,\psi\,(y)+\bar\psi\,(y)\,\zeta(y)\,\right]\right\}\right\rangle_0
\end{aligned}
$$

The vacuum expectation values of the chronological ordered products of n pairs of free spinor field and its adjoint operators at different spacetime points are named the $n-$point fermion Green's functions of the (free) Dirac spinor quantum field theory. By construction, the latter ones can be expressed as functional derivatives of the generating functional : namely,

$$
\begin{aligned}
&S_0^{(n)}(\,x_1,\cdots,x_n\,;\,y_1,\cdots,y_n\,) \;\stackrel{\mathrm{def}}{=}\\
&\langle 0|\,T\,\psi\,(x_1)\,\bar\psi\,(y_1)\,\cdots\,\psi\,(x_n)\,\bar\psi\,(y_n)\,|0\rangle\\
&=\;\delta^{\,(2n)}\,Z_0[\,\zeta\,,\,\bar\zeta\,]\,/\,\delta\bar\zeta(x_1)\,\delta\zeta(y_1)\,\cdots\,\delta\bar\zeta(x_n)\,\delta\zeta(y_n)\,\big]_{\zeta=\bar\zeta=0}
\end{aligned}
$$

Now it should be evident that the very same steps, which have led to establish the functional equation (3.101) for the free scalar field generating functional, can be repeated in a straightforward manner. To this purpose, let me introduce the finite times chronologically ordered exponential operator for the spinor fields that reads

$$E(t'\,,\,t) \equiv T\,\exp\left\{\mathrm{i}\int_t^{t'}\mathrm{d}y^0\int\mathrm{d}\mathbf{y}\;[\,\bar{\psi}(y)\,\zeta(y)+\bar{\zeta}(y)\,\psi(y)\,]\right\} \tag{4.134}$$

in such a manner that we can write

$$\begin{aligned}
&(\,\mathrm{i}\,\partial\!\!\!/_x - M\,)\Big(-\mathrm{i}\,\delta Z_0[\,\zeta\,,\,\bar{\zeta}\,]\,/\,\delta\bar{\zeta}\,(x)\Big) \;=\\
&\mathrm{i}\,\gamma^0\int\mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\Big[\,\psi\,(x)\,\bar{\psi}\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y})\\
&-\bar{\psi}\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y})\,\psi\,(x)\Big]\,E\,(x^0,-\infty)\,|\,0\,\rangle\\
&=\;\mathrm{i}\,\gamma^0\int\mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\{\psi\,(x)\,,\,\bar{\psi}\,(x^0,\mathbf{y})\}\,\zeta\,(x^0,\mathbf{y})\,E\,(x^0,-\infty)\,|\,0\,\rangle\\
&=\;-\,\zeta\,(x)\,Z_0[\,\zeta\,,\,\bar{\zeta}\,]
\end{aligned}$$

hence

$$\Big[\,(\,\mathrm{i}\,\partial\!\!\!/_x - M\,)\;\mathrm{i}\,\delta/\,\delta\bar{\zeta}\,(x) - \zeta\,(x)\Big]\,Z_0[\,\zeta\,,\,\bar{\zeta}\,]\;=0 \tag{4.135}$$

In a complete analogous way we find the functional differential formula

$$\Big[\,\mathrm{i}\,\delta/\,\delta\zeta\,(x)\Big(\,\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_x + M\Big)\;+\bar{\zeta}\,(x)\Big]\,Z_0[\,\zeta\,,\,\bar{\zeta}\,]\;=0 \tag{4.136}$$

`Proof.` First we obtain

$$\begin{aligned}
&-\mathrm{i}\,\delta Z_0[\,\zeta\,,\,\bar{\zeta}\,]\,/\,\delta\zeta\,(x) =\\
&\langle\,0\,|\,E\,(\infty,x_0)\,\bar{\psi}(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle
\end{aligned}$$

and taking left time derivative

$$\begin{aligned}
&\langle\,0\,|\,E\,(\infty,x_0)\,\bar{\psi}(x_0,\mathbf{x})\,E\,(x_0,-\infty)\,|\,0\,\rangle\,\frac{\overleftarrow{\partial}}{\partial x^0}\\
=\;&\mathrm{i}\int\mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)\,\bar{\psi}\,(x)\,[\,\bar{\psi}\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y})+\bar{\zeta}\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,]E\,(x^0,-\infty)\,|\,0\,\rangle\\
-\;&\mathrm{i}\int\mathrm{d}\mathbf{y}\;\langle\,0\,|\,E\,(\infty,x^0)[\,\bar{\psi}\,(x^0,\mathbf{y})\,\zeta\,(x^0,\mathbf{y})+\bar{\zeta}\,(x^0,\mathbf{y})\,\psi\,(x^0,\mathbf{y})\,]\,\bar{\psi}\,(x)\,E\,(x^0,-\infty)\,|\,0\,\rangle\\
+\;&\Big\langle T\,\partial_0\bar{\psi}\,(x)\exp\Big\{\mathrm{i}\int\mathrm{d}y\;\Big[\,\bar{\zeta}(y)\,\psi\,(y)+\bar{\psi}\,(y)\,\zeta(y)\Big]\Big\}\Big\rangle_0
\end{aligned}$$

Now, using the anticommutation relations (4.128), (4.129) and the equal time canonical anticommutation relations (4.60) we find

$$
\begin{aligned}
& \langle\, 0 \,|\, E\,(\infty, x_0)\, \bar\psi(x_0, \mathbf{x})\, E\,(x_0, -\infty)\,|\, 0\,\rangle\,(\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_x + M\,) \\
= \;& \gamma^0 \int \mathrm{d}\mathbf{y}\; \langle\, 0 \,|\, E\,(\infty, x_0)\, \bar\zeta\,(x_0, \mathbf{y}) \{ \bar\psi\,(x_0, \mathbf{x})\,,\, \psi\,(x^0, \mathbf{y}) \} E\,(x_0, -\infty)\,|\, 0\,\rangle \\
- \;& \gamma^0 \int \mathrm{d}\mathbf{y}\; \langle\, 0 \,|\, E\,(\infty, x_0) \{ \bar\psi\,(x_0, \mathbf{x})\,,\, \bar\psi\,(x_0, \mathbf{y}) \}\, \zeta\,(x_0, \mathbf{y})\, E\,(x_0, -\infty)\,|\, 0\,\rangle \\
+ \;& \left\langle T\,\bar\psi\,(x)(\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_x + M\,) \exp\left\{ \mathrm{i} \int \mathrm{d}y\, \left[\bar\zeta(y)\, \psi\,(y) + \bar\psi\,(y)\, \zeta(y) \right] \right\} \right\rangle_0 \\
= \;& \int \mathrm{d}\mathbf{y}\; \langle\, 0 \,|\, E\,(\infty, x_0)\, \bar\zeta\,(x_0, \mathbf{y})\, \delta(\mathbf{x} - \mathbf{y})\, E\,(x_0, -\infty)\,|\, 0\,\rangle = \bar\zeta\,(x)\, Z_0\,[\,\zeta\,,\,\bar\zeta\,]
\end{aligned}
$$

where use has been made of the adjoint Dirac equation $\bar\psi(x)(\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_x + M\,) = 0$. This proves the Symanzik equation (4.136). *Quod Erat Demonstrandum*

The solution of the above couple of functional differential equations that satisfies causality is

$$
Z_0\,[\,\zeta\,,\,\bar\zeta\,] = \exp\left\{ -\int \mathrm{d}\,x \int \mathrm{d}\,y\; \bar\zeta\,(x)\, S_F\,(x-y)\, \zeta\,(y) \right\} \tag{4.137}
$$

It is worthwhile to remark that, in close analogy with the case of the free scalar field generating functional, the integral kernel $-\,S_F\;(x-y)$ which appears in the exponent of the right hand side of eq. (4.137) does exactly coincide with the inverse of the Dirac operator $(\mathrm{i}\,\partial\!\!\!/ - M)$ that specify the classical spinor action , since $(M - \mathrm{i}\,\partial\!\!\!/_x)\; S_F\,(x{-}y) = \delta\,(x-y)$. The inversion of the classical kinetic Dirac operator is precisely provided by the Feynman propagator or causal 2-point Green's function, which indeed encodes :

1. the relativistic covariance properties under the action of the Poincaré group

2. the canonical anticommutation relations (4.64), the related property (4.97) of the microcausality and the Fermi-Dirac statistics for the multi-particle states

3. the causality requirement, that means the possibility to perform the Wick rotation and turn smoothly to the euclidean formulation.

Hence the generating functional (4.137) does truly contain all the mutually tied up key features of the relativistic quantum field theory.

4.6.2 The Integration over Grassmann Variables

In order to find a functional integral representation for $Z_0\,[\,\zeta\,,\,\bar{\zeta}\,]$ I need to primary define the concept of integration with respect to Grassmann valued functions on the Minkowski spacetime. This latter tool has been long ago constructed by

F.A. Berezin, *The Method of Second Quantization*
Academic Press, New York (1966)

Hereafter I shall follow

Sidney R. Coleman
The Uses of Instantons
Proceedings of the 1977 International School of Subnuclear Physics
Erice, Antonino Zichichi Editor, Academic Press, New York (1979)

Consider first a real function $f\,:\,\mathfrak{G}\,\to\,\mathbb{R}$ of a Grassmann variable a with $a^2\,=\,0$: we want to define $\int da\, f(a)$. We require this to have the usual linearity property :

$$\begin{aligned}&\int da\;[\,\alpha f(a)+\beta\, g\,(a)\,]\\ =\;\;&\alpha\int da\;f(a)+\beta\int da\;g\,(a)\qquad\quad(\forall\,\alpha,\beta\in\mathbb{R}\,)\end{aligned}\tag{4.138}$$

In addition, we would like the integral to be translation invariant

$$\int da\;f(a+b)=\int da\;f(a)\qquad\quad(\forall\,b\in\mathfrak{G}\,)\tag{4.139}$$

It is easy to show that those conditions determine the integral, up to a normalization constant.

The reason is that there are only two linearly independent functions of $a\in\mathfrak{G}$, that is 1 and a , any higher powers being vanishing. As a matter of fact we have $\forall\,a,b\in\mathfrak{G}$ with $\{a,a\}=\{a,b\}=\{b,b\}=0$

$$f(a)=f(0)+f\,'(0)\,a\qquad\qquad f(a+b)=f(0)+f\,'(0)\,(a+b)\tag{4.140}$$

whence from (4.138)

$$\begin{aligned}\int da\;f(a)\;&=\;\int da\;f(0)+\int da\;f\,'(0)\,a\\ &=\;f(0)\int da\;1+f\,'(0)\int da\;a\\ \int da\;f(a+b)\;&=\;\int da\;f(0)+\int da\;f\,'(0)\,(a+b)\\ &=\;f(0)\int da\;1+f\,'(0)\int da\;a+b\,f\,'(0)\int da\;1\end{aligned}$$

and from the translation invariance requirement (4.139)

$$\int da\ a = N \qquad b\,f'(0)\int da\ 1 \equiv 0 \qquad (\,\forall\, b \in \mathfrak{G}\,) \tag{4.141}$$

One can always choose the normalization constant N such that

$$\int da\ a = 1$$

But then, translation invariance just requires

$$\int da\ 1 = 0$$

Hence

$$\int da\ f(a) = f'(0) \qquad \forall\, f\ :\ \mathfrak{G} \to \mathbb{R}$$

As a straightforward generalization for any many Grassmann variables function, it is natural to define multiple integrals by iteration. Thus, *e.g.* , a complete integration table for the four linearly independent functions of two anticommuting variables a and $\bar a$ is provided by

$$\int da \int d\bar a \begin{cases} \bar a\, a &= 1 \\ \bar a &= 0 \\ a &= 0 \\ 1 &= 0 \end{cases} \tag{4.142}$$

with $\{a, a\} = \{\bar a, \bar a\} = \{a, \bar a\} = 0$. As an application of this table we can calculate

$$\int da \int d\bar a\ \exp\{\,\lambda\, \bar a\, a\} = \int da \int d\bar a\ (1 + \lambda\, \bar a\, a) = \lambda \tag{4.143}$$

The generalization to a multi-dimensional space is straightforward. Consider some $n \times n$ hermitean matrix $A = A^\dagger$ and two sets of n Grassmann variables

$$\{\theta_i\,,\,\theta_j\} = \{\theta_i\,,\,\bar\theta_j\} = \{\bar\theta_i\,,\,\bar\theta_j\} = 0 \qquad (\,i, j = 1, 2, \ldots, n\,)$$

Then a unitary $n \times n$ matrix always exists such that

$$U^\dagger A U = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \qquad \lambda_j \in \mathbb{R} \quad (\,j = 1, 2, \ldots, n\,)$$

As a consequence, if we set

$$a = U\,\theta \qquad \bar a = \bar\theta\, U^\dagger$$

so that

$$\{a_i\,,\,a_j\} = \{a_i\,,\,\bar{a}_j\} = \{\bar{a}_i\,,\,\bar{a}_j\} = 0 \qquad (i,j = 1,2,\ldots,n) \tag{4.144}$$

then we can write

$$\begin{aligned}
I &= \int da \int d\bar{a}\ \exp\{\bar{a}\,A\,a\} \\
&= \det(U^{\dagger}U) \int d\theta \int d\bar{\theta}\ \exp\{\bar{\theta}\,U^{\dagger}\,A U\,\theta\} \\
&= \prod_{j=1}^{n} \int d\theta_j \int d\bar{\theta}_j\ \exp\{\,\lambda_j\,\bar{\theta}_j\,\theta_j\} \\
&= \prod_{j=1}^{n} \int d\theta_j \int d\bar{\theta}_j\ \left(1 + \lambda_j\,\bar{\theta}_j\,\theta_j\right) \\
&= \prod_{j=1}^{n} \lambda_j \ = \ \det A
\end{aligned} \tag{4.145}$$

As a further application, let us consider the Taylor expansion for a real function of the Grassmann variables $(\bar{a}, a)$ satisfying (4.144): namely,

$$\begin{aligned}
f(\bar{a},a) &= f(0,0) + \sum_{\jmath=1}^{n} \bar{a}_\jmath \left(\frac{\partial f}{\partial \bar{a}_\jmath}\right)_0 + \sum_{\imath=1}^{n} a_\imath \left(\frac{\partial f}{\partial a_\imath}\right)_0 \\
&+ \ \tfrac{1}{2} \sum_{\imath=1}^{n} \sum_{\jmath=1}^{n} \bar{a}_\jmath\,a_\imath \left(\frac{\partial^2 f}{\partial a_\imath\,\partial \bar{a}_\jmath}\right)_0
\end{aligned}$$

where

$$\left(\frac{\partial f}{\partial \bar{a}_\jmath}\right)_0 \equiv \frac{\partial}{\partial \bar{a}_\jmath}\,f(\bar{a},a)\bigg|_{\bar{a}=a=0} \qquad \text{et cetera}$$

Then we have *e.g.*

$$\begin{aligned}
&\int da_\jmath \int d\bar{a}_\jmath\,\frac{\partial}{\partial \bar{a}_\jmath}\,f(\bar{a},a) \ = \\
&\left(\frac{\partial f}{\partial \bar{a}_\jmath}\right)_0 \int da_\jmath \int d\bar{a}_\jmath - \frac{1}{2}\left(\frac{\partial^2 f}{\partial a_\jmath\,\partial \bar{a}_\jmath}\right)_0 \int da_\jmath\,a_\jmath \int d\bar{a}_\jmath \equiv 0
\end{aligned}$$

and in general

$$\int da \int d\bar{a}\,[\,\partial f(\bar{a},a)/\partial \bar{a}_\jmath\,] = 0 \qquad (\,\forall\,\jmath = 1,2,\ldots,n\,) \tag{4.146}$$

$$\int da \int d\bar{a}\,[\,\partial f(\bar{a},a)/\partial a_\imath\,] = 0 \qquad (\,\forall\,\imath = 1,2,\ldots,n\,) \tag{4.147}$$

4.6.3 The Functional Integrals for Fermions

Consider now the straightforward generalization of the functional integral representation (3.105) to the fermion case : namely,

$$\begin{aligned} Z_0[\zeta\,,\bar\zeta\,] \;&=\; \int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\;\widetilde{Z}_0[\,\bar\psi\,,\,\psi\,] \\ &\times\;\;\exp\left\{\mathrm{i}\int\mathrm{d}x\;\Big[\,\bar\zeta(x)\,\psi\,(x)+\bar\psi\,(x)\,\zeta(x)\Big]\right\} \qquad (4.148)\end{aligned}$$

where we understand once again formally

$$\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\;:=:\;\prod_{x\in\mathcal{M}}\int\mathrm{d}\,\psi_x\int\mathrm{d}\,\bar\psi_x$$

Inserting into (4.135) and (4.136) and making use of (4.146) and (4.147) we are enabled to identify

$$\widetilde{Z}_0[\,\bar\psi\,,\,\psi\,]=\mathcal{N}\,\exp\,\{\,\mathrm{i}\,S_0[\,\bar\psi\,,\,\psi\,]\} \qquad (4.149)$$

and by comparison with (4.137) we find

$$\begin{aligned} Z_0[\,\zeta\,,\bar\zeta\,] \;&=\;\exp\left\{-\int\mathrm{d}x\int\mathrm{d}y\;\bar\zeta\,(x)\,S_F\,(x-y)\,\zeta\,(y)\right\} \\ &:=:\;\mathcal{N}\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\;\exp\,\{\,\mathrm{i}\,S_0[\,\bar\psi\,,\,\psi\,]\} \\ &\times\;\;\exp\left\{\mathrm{i}\int\mathrm{d}x\;\Big[\,\bar\zeta(x)\,\psi\,(x)+\bar\psi\,(x)\,\zeta(x)\Big]\right\} \qquad (4.150)\end{aligned}$$

$$S_0[\,\bar\psi\,,\,\psi\,]=\int\mathrm{d}x\;\bar\psi\,(x)\,(\mathrm{i}\,\partial\!\!\!/-M)\;\psi\,(x)$$

As I did in the case of the scalar field, the functional measure $\mathfrak{D}\psi\,\mathfrak{D}\bar\psi$, which is a formal entity until now, can be implemented by the requirement of invariance under field translations $\psi(x)\;\mapsto\;\psi(x)+\theta(x)$, $\bar\psi(x)\;\mapsto\;\bar\psi(x)+\bar\theta(x)$. Once it is assumed, after the change of variables

$$\begin{aligned} \psi\,(x)\;\mapsto\;\psi\,'(x)\;&=\;\psi\,(x)+(\mathrm{i}\,\partial\!\!\!/-M)^{-1}\,\zeta\,(x) \\ &=\;\psi\,(x)-\mathrm{i}\textstyle\int\mathrm{d}y\,S_F(x-y)\,\zeta\,(y) \\ &=\;\psi_x-\langle\,\mathrm{i}\,S_{xy}\,\zeta_y\,\rangle \qquad (4.151) \\ \bar\psi\,(x)\;\mapsto\;\bar\psi\,'(x)\;&=\;\bar\psi\,(x)+\bar\zeta\,(x)(\mathrm{i}\,\overleftarrow{\partial\!\!\!/}+M\,)^{-1} \\ &=\;\bar\psi\,(x)-\mathrm{i}\textstyle\int\mathrm{d}y\,\bar\zeta\,(y)\,\bar S_F(y-x) \\ &=\;\bar\psi_x-\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle \qquad (4.152)\end{aligned}$$

where use has been made of the equation (4.105). Then we readily obtain

$$\begin{aligned} Z_0[\zeta\,,\bar\zeta\,] &:=: Z_0[\,0\,,\,0\,]\,\exp\left\{-\mathrm{i}\int\mathrm{d}x\int\mathrm{d}y\;\bar\zeta(x)\,(\mathrm{i}\partial\!\!\!/-M)^{-1}\,\zeta(y)\right\}\\ &= Z_0[\,0\,,\,0\,]\,\exp\left\{-\int\mathrm{d}x\int\mathrm{d}y\;\bar\zeta(x)\,S_F(x-y)\,\zeta(y)\right\}\end{aligned}$$

so that consistency requires

$$Z_0[\,0\,,\,0\,] = \langle\,0\,|\,0\,\rangle :=: \mathcal{N}\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\;\exp\left\{\,\mathrm{i}\,S_0[\,\bar\psi\,,\,\psi\,]\,\right\}\;=\;1$$

`Proof` : the formal expression (4.150) can be suitably rewritten as

$$Z_0[\,\zeta\,,\bar\zeta\,] :=: \mathcal{N}\prod_{x\in\mathcal{M}}\int\mathrm{d}\psi_x\int\mathrm{d}\bar\psi_x\;\exp\left\{\mathrm{i}\,\bar\psi_x\,(\mathrm{i}\partial\!\!\!/_x-M)\,\psi_x+\mathrm{i}\,\bar\psi_x\,\zeta_x+\mathrm{i}\,\bar\zeta_x\,\psi_x\right\}$$

Then, once again, it is very convenient to perform the change of variables

$$\begin{aligned} \psi_x &= \psi'_x+\langle\,\mathrm{i}\,S_{xy}\,\zeta_y\,\rangle &\qquad \mathrm{d}\psi_x &= \mathrm{d}\psi'_x\\ \bar\psi_x &= \bar\psi'_x+\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle &\qquad \mathrm{d}\bar\psi_x &= \mathrm{d}\bar\psi'_x\end{aligned}$$

in such a manner that we have

$$\begin{aligned} &\int\mathrm{d}\psi_x\int\mathrm{d}\bar\psi_x\,\exp\left\{\mathrm{i}\,\bar\psi_x\,(\mathrm{i}\partial\!\!\!/_x-M)\,\psi_x+\mathrm{i}\,\bar\psi_x\,\zeta_x+\mathrm{i}\,\bar\zeta_x\,\psi_x\right\}\\ =\;&\int\mathrm{d}\psi_x\int\mathrm{d}\bar\psi_x\,\exp\left\{\mathrm{i}\left(\bar\psi'_x+\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle\right)\,(\mathrm{i}\partial\!\!\!/_x-M)\,\left(\psi'_x+\langle\,\mathrm{i}\,S_{xz}\,\zeta_z\,\rangle\right)\right\}\\ \times\;&\exp\left\{\,\mathrm{i}\,\bar\psi'_x\,\zeta_x+\mathrm{i}\,\bar\zeta_x\,\psi'_x-\langle\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,\zeta_x-\bar\zeta_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\,\right\}\end{aligned}$$

and from the first equality

$$\bar\psi'_x\,(\mathrm{i}\partial\!\!\!/_x-M)\,\langle\,\mathrm{i}\,S_{xz}\,\zeta_z\,\rangle\;=\;-\,\bar\psi'_x\,\zeta_x$$

together with the second conditioned equality

$$\begin{aligned} &\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(\mathrm{i}\partial\!\!\!/_x-M)\,\psi'_x\\ =\;&-\,(\,\partial/\partial x^{\mu}\,)\,\left[\,\langle\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,\gamma^{\mu}\,\psi'_x\,\right]-\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(\,\mathrm{i}\overleftarrow{\partial\!\!\!/}_x+M\,)\,\psi'_x\\ \doteq\;&-\,\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,(\,\mathrm{i}\overleftarrow{\partial\!\!\!/}_x+M\,)\,\psi'_x\;=\;-\,\bar\zeta_x\,\psi'_x\end{aligned}$$

which is true by neglecting the four divergence term, we can eventually write

$$\begin{aligned} &\exp\left\{\mathrm{i}\left(\bar\psi'_x+\langle\,\mathrm{i}\,\bar\zeta_y\,\bar S_{yx}\,\rangle\right)\,(\mathrm{i}\partial\!\!\!/_x-M)\,\left(\psi'_x+\langle\,\mathrm{i}\,S_{xz}\,\zeta_z\,\rangle\right)\right\}\\ \doteq\;&\exp\left\{\,\mathrm{i}\,\bar\psi'_x\,(\,\mathrm{i}\partial\!\!\!/_x-M\,)\,\psi'_x-\mathrm{i}\,\bar\psi'_x\,\zeta_x-\,\mathrm{i}\,\bar\zeta_x\,\psi'_x+\langle\,\bar\zeta_y\,\bar S_{yx}\,\rangle\,\zeta_x\,\right\}\end{aligned}$$

Hence, collecting altogether we come to the double integral

$$\begin{aligned} &\int\mathrm{d}\psi_x\int\mathrm{d}\bar\psi_x\,\exp\left\{\mathrm{i}\,\bar\psi_x\,(\mathrm{i}\partial\!\!\!/_x-M)\,\psi_x+\mathrm{i}\,\bar\psi_x\,\zeta_x+\mathrm{i}\,\bar\zeta_x\,\psi_x\right\}\\ \doteq\;&\exp\left\{-\,\bar\zeta_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\,\right\}\int\mathrm{d}\psi'_x\int\mathrm{d}\bar\psi'_x\,\exp\left\{\mathrm{i}\,\bar\psi'_x\,(\mathrm{i}\partial\!\!\!/_x-M)\,\psi'_x\right\}\qquad(\forall x\in\mathcal{M}\,)\end{aligned}$$

and thereby

$$
\begin{aligned}
& \mathcal{N} \prod_{x\,\in\,\mathcal{M}} \int \mathrm{d}\psi_x \int \mathrm{d}\bar{\psi}_x \, \exp\left\{\mathrm{i}\,\bar{\psi}_x\,(\mathrm{i}\,\partial\!\!\!/_x - M)\,\psi_x + \mathrm{i}\,\bar{\psi}_x\,\zeta_x + \mathrm{i}\,\bar{\zeta}_x\,\psi_x\right\} \\
\doteq\ & \prod_{x\,\in\,\mathcal{M}} \exp\left\{\,-\bar{\zeta}_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\,\right\}\ \mathcal{N} \prod_{z\,\in\,\mathcal{M}} \int \mathrm{d}\psi'_z \int \mathrm{d}\bar{\psi}'_z \, \exp\left\{\mathrm{i}\,\bar{\psi}'_z\,(\mathrm{i}\,\partial\!\!\!/_z - M)\,\psi'_z\right\} \\
=\ & \prod_{x\,\in\,\mathcal{M}} \exp\left\{-\bar{\zeta}_x\,\langle\,S_{xy}\,\zeta_y\,\rangle\right\}\ \mathcal{N} \int \mathfrak{D}\psi' \int \mathfrak{D}\bar{\psi}' \ \exp\left\{\mathrm{i}\,S_0\,[\,\bar{\psi}'\,,\,\psi'\,]\right\} \\
=\ & \exp\left\{-\int \mathrm{d}x \int \mathrm{d}y\ \bar{\zeta}(x)\,S_F\,(x-y)\,\zeta(y)\right\}\ Z_0\,[\,0\,,\,0\,] = Z_0\,[\,\zeta\,,\,\bar{\zeta}\,]
\end{aligned}
$$

Quod Erat Demonstrandum

To the aim of giving some precise mathematical meaning to the above expression, it is convenient to turn to the euclidean formulation just like I did in the case of the real scalar field – see formula (3.113). Then, taking the Dirac euclidean action (4.115) into account, we can consider

$$
\begin{aligned}
Z_E^{(0)}\,[\,0\,,\,0\,] \ &= \ \mathcal{N} \int \mathfrak{D}\psi_E \int \mathfrak{D}\bar{\psi}_E \\
&\times \ \exp\left\{\mathrm{i} \int \mathrm{d}x_E \ \bar{\psi}_E\,(\mathrm{i}\,\partial\!\!\!/_E + iM)\ \psi_E\right\}
\end{aligned}
$$

After rescaling

$$
\psi'_{E\,,\,x} = \mathrm{i}\,\mu\,\psi_{E\,,\,x}
$$

where μ is some arbitrary mass scale, the we can write

$$
\begin{aligned}
& Z_E^{(0)}\,[\,0\,,\,0\,] \ = \\
& \mathcal{N}' \prod_{x_E\,\in\,\mathbb{R}^4} \int \mathrm{d}\psi'_{Ex} \int \mathrm{d}\bar{\psi}'_{Ex} \, \exp\left\{\,\bar{\psi}'_{Ex}\,(\mathrm{i}\,\partial\!\!\!/_{Ex} + \mathrm{i}M)\,\mu^{-1}\,\psi'_{Ex}\right\} \\
& \stackrel{\mathrm{def}}{=} \mathcal{N}'\,\det\ \|\ (\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)/\mu\ \|
\end{aligned}
\tag{4.153}
$$

which can be understood as a formal continuous generalization of the formula (4.145) that represents a determinant. Nevertheless, it should be noticed that the euclidean free Dirac operator $(\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)$ is not hermitean but only normal, *i.e.* , it commutes with its adjoint $[\,\mathrm{i}\,\partial\!\!\!/_E - \mathrm{i}M\,,\,\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M\,] = 0\,.$

This fact implies that the euclidean free Dirac operator $\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M$ also commutes with the *positive definite* diagonal operator

$$
(\mathrm{i}\,\partial\!\!\!/_E - \mathrm{i}M)(\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M) = \left(M^2 - \partial_{E\mu}\,\partial_{E\mu}\right)\,\mathbf{I} = (\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)(\mathrm{i}\,\partial\!\!\!/_E - \mathrm{i}M)
$$

that means

$$
[\,\mathrm{i}\,\partial\!\!\!/_E \pm \mathrm{i}M\,,\,M^2 - \partial_{E\mu}\,\partial_{E\mu}\,] = 0
$$

where the eigenfunctions and the eigenvalues of the euclidean Klein-Gordon operator are respectively given by

$$\phi_{E\,,p}\,(x_E) = (2\pi)^{-2}\,\exp\{\,\mathrm{i}p_{E\mu}x_{E\mu}\} \qquad \lambda\,(p_E) = p_E^2 + M^2 > 0$$

Notice that the normal operator $\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M$ is also invertible, since we have

$$(\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)^{-1} = (\mathrm{i}\,\partial\!\!\!/_E - \mathrm{i}M)\left(M^2 - \partial_{E\mu}\,\partial_{E\mu}\right)^{-1}$$

From the explicit form

$$\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M = \left(\begin{array}{cccc} \mathrm{i}M & 0 & \mathrm{i}\partial_4 + \partial_3 & \partial_1 - \mathrm{i}\partial_2 \\ 0 & \mathrm{i}M & \partial_1 + \mathrm{i}\partial_2 & \mathrm{i}\partial_4 - \partial_3 \\ \mathrm{i}\partial_4 - \partial_3 & -\partial_1 + \mathrm{i}\partial_2 & \mathrm{i}M & 0 \\ -\partial_1 - \mathrm{i}\partial_2 & \mathrm{i}\partial_4 + \partial_3 & 0 & \mathrm{i}M \end{array}\right)$$

it can be readily checked by direct inspection that

$$\det\ \|\ (\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)/\mu\ \| = \left(\det\ \|\ \left(M^2 - \partial_E^2\right)/\mu^2\ \|\ \right)^2$$

so that from the Zeta function definition (3.123) we eventually obtain

$$\det\ \|\ (\mathrm{i}\,\partial\!\!\!/_E + \mathrm{i}M)\,/\mu\ \| = \ \exp\left\{\frac{VM^4}{8\pi^2}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\} \tag{4.154}$$

Turning back to (4.153) we have

$$Z_E^{(0)}\,[\,0\,,\,0\,] = 1 \quad \Longleftrightarrow \quad \mathcal{N}' \equiv \exp\left\{-\,\frac{VM^4}{8\pi^2}\left(\ln\frac{M}{\mu} - \frac{3}{4}\right)\right\}$$

and again the transition to the Minkowski spacetime can be immediately done by simply replacing $V_{\mathrm{euclidean}} \quad \leftrightarrow \quad \mathrm{i}\,V_{\mathrm{minkowskian}}$

The conclusion of all the above formal reasoning is as follows : we are allowed to *define the functional integral* for a free Dirac spinor quantum field theory by the equalities

$$\begin{aligned} Z_0\,[\,\zeta\,,\,\bar\zeta\,] \;&=\; \exp\left\{\,-\int\mathrm{d}\,x\int\mathrm{d}\,y\ \bar\zeta\,(x)\,S_F\,(x-y)\,\zeta\,(y)\right\} \\ &\stackrel{\mathrm{def}}{=}\; \mathcal{N}\int\mathfrak{D}\psi\int\mathfrak{D}\bar\psi\ \exp\left\{\,\mathrm{i}\,S_0\,[\,\bar\psi\,,\,\psi\,]\right\} \\ &\times\; \exp\left\{\mathrm{i}\int\mathrm{d}\,x\,\left[\,\bar\zeta(x)\,\psi\,(x) + \bar\psi\,(x)\,\zeta(x)\right]\right\} \end{aligned} \tag{4.155}$$

where

$$S_0[\,\bar{\psi}\,,\,\psi\,] = \int \mathrm{d}x\;\bar{\psi}(x)\,(\mathrm{i}\,\partial\!\!\!/ - M)\;\psi(x)$$

$$\begin{aligned}
\mathcal{N} \quad &= \quad \text{constant} \times \det\;\|\;\mathrm{i}\,\partial\!\!\!/ - M\;\| \\
&\stackrel{\text{def}}{=} \quad \exp\left\{-\,\mathrm{i}\,\frac{VM^4}{8\pi^2}\left(\ln\frac{M}{\mu}-\frac{3}{4}\right)\right\} \qquad \text{(Zeta regularization)}
\end{aligned}$$

$$Z_0[\,0\,,\,0\,] = \mathcal{N}\int\mathfrak{D}\psi\int\mathfrak{D}\bar{\psi}\;\exp\left\{\,\mathrm{i}\,S_0[\,\bar{\psi}\,,\,\psi\,]\right\} = 1 = \langle\,0\,|\,0\,\rangle$$

It is clear that, by construction, the functional integral for a free Dirac spinor quantum field theory does satisfy all the requirements of linearity, translation invariance, rescaling and integration by parts which I have discussed in the case of the real scalar field.

4.7 The $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ Transformations

The charge conjugation $\mathcal{C}$ is the discrete internal symmetry transformation under which the particles and antiparticles are interchanged. The parity transformation $\mathcal{P}$ or spatial inversion is the discrete spacetime symmetry transformation such that

$$x^{\mu} = (t, \mathbf{r}) \mapsto x'^{\mu} = (t, -\mathbf{r}) = \mathcal{P}\, x^{\mu}$$

Under $\mathcal{P}$ the handedness of the particles motion is reversed so that, for example, a left handed electron e_L^- is transformed into a right handed positron e_R^+ under the combined $\mathcal{CP}$ symmetry transformation. Thus, if $\mathcal{CP}$ were an exact symmetry, the laws of Nature would be the same for matter and for antimatter.

The experimental evidence actually shows that most phenomena in the particles Physics are $\mathcal{C}$ and $\mathcal{P}$ symmetric and thereby also $\mathcal{CP}$ symmetric. In particular, these symmetries are respected by the electromagnetic and strong interactions as well as by classical gravity. On the other hand, the weak interactions violate $\mathcal{C}$ and $\mathcal{P}$ in the strongest possible way. Hence, while weak interactions do violate $\mathcal{C}$ and $\mathcal{P}$ symmetries separately, the combined $\mathcal{CP}$ symmetry is still preserved. The $\mathcal{CP}$ symmetry is, however, violated in certain rare processes, as discovered long ago in neutral K mesons decays and recently observed in neutral B decays. Thus, only the combined discrete $\mathcal{CPT}$ symmetry transformation, where $\mathcal{T}$ denotes the time inversion

$$x^{\mu} = (t, \mathbf{r}) \mapsto x'^{\mu} = (-\,t, \mathbf{r}) = \mathcal{T}\, x^{\mu}$$

is an exact symmetry for all laws of Nature, just like for the invariance under the restricted Poincaré continuous group. In the sequel we will examine in some detail the $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations for the quantized Dirac field.

4.7.1 The Charge Conjugation

We have seen before that the quantization of the free Dirac wave field leads to the appearance of particle and antiparticles, which are characterized by the very same mass and spin although opposite charge. The latter one may have a different interpretation depending upon the physical context: namely, it could be electric, barionic, leptonic *et cetera*. In any case, the existence of the charge conjugation invariance just implies

1. the existence of the antiparticles

2. the equality of all the quantum numbers, but charge, for a particle antiparticle pair

We are looking for a unitary charge conjugation operator in the theory of the quantized Dirac field, the action of which will be given by

$$\psi^{\,c}(x) = \mathcal{C}\,\psi(x)\,\mathcal{C}^{\,\dagger} \tag{4.156}$$

Charge conjugation is conventionally defined as the operation in which particles and antiparticles are interchanged. It follows thereby that if we set

$$\mathcal{C}\,c_{\mathbf{p}\,,r}\,\mathcal{C} = d_{\mathbf{p}\,,s} \qquad \mathcal{C}\,d_{\mathbf{p}\,,r}\,\mathcal{C} = c_{\mathbf{p}\,,s} \tag{4.157}$$

$$\forall\,r,s = 1,2 \quad \mathbf{p}\in\mathbb{R}^3$$

with

$$\mathcal{C}^{\,2} \;=\; \mathbf{I} \qquad\qquad \mathcal{C} = \mathcal{C}^{\,\dagger} = \mathcal{C}^{\,-1} \tag{4.158}$$

It turns out that the standard spin-states (4.33) do fulfill the remarkable relationship

$$\begin{aligned} u_r(\mathbf{p}) \;&=\; -\,\mathrm{i}\,\gamma^2\,v_r^*(\mathbf{p}) \\ v_r(\mathbf{p}) \;&=\; -\,\mathrm{i}\,\gamma^2\,u_r^*(\mathbf{p}) \end{aligned}$$

Hence, the transformation law (4.156) can be rewritten as

$$\begin{aligned} \psi^{\,c}(x) \;&=\; \sum_{\mathbf{p}\,,r}\;\Big[\,d_{\mathbf{p}\,,r}\,u_{\mathbf{p}\,,r}(x) + c^{\,\dagger}_{\mathbf{p}\,,r}\,v_{\mathbf{p}\,,r}(x)\,\Big] \\ &=\; \frac{1}{\mathrm{i}}\sum_{\mathbf{p}\,,r}\;\Big[\,d_{\mathbf{p}\,,r}\,\gamma^2\,v^*_{\mathbf{p}\,,r}(x) + c^{\,\dagger}_{\mathbf{p}\,,r}\,\gamma^2\,u^*_{\mathbf{p}\,,r}(x)\,\Big] \\ &=\; -\,\mathrm{i}\,\gamma^2\,\left(\psi^{\,\dagger}(x)\right)^{\top} \;=\; \left(-\,\mathrm{i}\,\bar{\psi}(x)\,\gamma^0\,\gamma^2\right)^{\top} \end{aligned} \tag{4.159}$$

which corresponds to the quantum mechanical counterpart of the classical transformation (2.81). Moreover we readily find

$$\bar{\psi}^{\,c}(x) = \psi^{\,c\,\dagger}(x)\,\gamma^0 = \left(-\,\mathrm{i}\,\gamma^2\,\psi(x)\right)^{\top}\,\gamma^0 = \left(-\,\mathrm{i}\,\gamma^0\,\gamma^2\,\psi(x)\right)^{\top} \tag{4.160}$$

Working out the transformations of bilinears is a little bit tricky and it helps to write the spinor indices explicitly. For instance, the mass scalar operator becomes

$$\begin{aligned} :\bar{\psi}^{\,c}(x)\,\psi^{\,c}(x): \;&=\; :\left(-\,\mathrm{i}\,\gamma^0\,\gamma^2\,\psi(x)\right)^{\top}\,\left(-\,\mathrm{i}\,\bar{\psi}(x)\,\gamma^0\,\gamma^2\right)^{\top}: \\ &=\; -\;:\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x)\;\bar{\psi}_\eta(x)\,\gamma^0_{\eta\omega}\,\gamma^2_{\omega\alpha}: \\ &=\; +\;:\bar{\psi}_\eta(x)\,\gamma^0_{\eta\omega}\,\gamma^2_{\omega\alpha}\;\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x): \\ &=\; +\;:\bar{\psi}(x)\,\gamma^0\,\gamma^2\,\gamma^0\,\gamma^2\,\psi(x): \\ &=\; -\;:\bar{\psi}(x)\,(\gamma^0)^2(\gamma^2)^2\,\psi(x): \\ &=\; +\;:\bar{\psi}(x)\,\psi(x): \end{aligned} \tag{4.161}$$

where the change of sign in the third line is just owing to the spinor field canonical anticommuation relations. Hence the femion mass scalar operator is invariant under charge conjugation, *i.e.*

$$:\bar{\psi}^{\,c}(x)\,\psi^{\,c}(x): \quad = \quad :\bar{\psi}(x)\,\psi(x):$$

while, in a quite analogous way, one can show that the electric current density changes its sign : namely,

$$:\bar{\psi}^{\,c}(x)\gamma^{\,\mu}\psi^{\,c}(x): \quad = \quad -\ :\bar{\psi}(x)\gamma^{\,\mu}\psi(x):$$

4.7.2 The Parity Transformation

Suppose it is possible, with a suitable modification of some experimental apparatus, to realize the space inversion and to obtain the parity transformed state of a Dirac particle or antiparticle. From the quantized Dirac field point of view we look for a unitary operator $\mathcal{P}$ satisfying

$$\begin{aligned} \psi'(x') &= \psi'(x^0,-\mathbf{x}) = \mathcal{P}\,\psi(x)\,\mathcal{P}^{\dagger} \\ &= \mathrm{e}^{\,\mathrm{i}\eta}\,\gamma^0\,\psi(x^0,-\mathbf{x}) \qquad\qquad (0\le\eta<2\pi) \end{aligned} \tag{4.162}$$

where the transformation law (2.59) of the classical Dirac wave field under space inversion has been taken into account. Inserting the normal mode expansion (4.28) we come to the relations

$$\begin{aligned} \mathcal{P}\,\psi(x)\,\mathcal{P}^{\dagger} &= \sum_{\mathbf{p},r}\Big[\mathcal{P}\,c_{\mathbf{p},r}\,\mathcal{P}^{\dagger}\,u_{\mathbf{p},r}(x)+\mathcal{P}\,d^{\dagger}_{\mathbf{p},r}\,\mathcal{P}^{\dagger}\,v_{\mathbf{p},r}(x)\Big] \\ &= \mathrm{e}^{\mathrm{i}\eta}\sum_{\mathbf{p},r}\Big[c_{\mathbf{p},r}\,\gamma^0\,u_{\mathbf{p},r}(t,-\mathbf{x})+d^{\dagger}_{\mathbf{p},r}\,\gamma^0\,v_{\mathbf{p},r}(t,-\mathbf{x})\Big] \end{aligned}$$

Now, it turns out that the previously introduced standard spin-states (4.33) do satisfy

$$\gamma^0\,u_r(-\,\mathbf{p}) = u_r(\mathbf{p}) \qquad \gamma^0\,v_r(-\,\mathbf{p}) = -\,v_r(\mathbf{p})$$

so that we can write

$$\begin{aligned} \mathcal{P}\,\psi(x)\,\mathcal{P}^{\dagger} &= \sum_{\mathbf{p},r}\Big[\mathcal{P}\,c_{\mathbf{p},r}\,\mathcal{P}^{\dagger}\,u_{\mathbf{p},r}(x)+\mathcal{P}\,d^{\dagger}_{\mathbf{p},r}\,\mathcal{P}^{\dagger}\,v_{\mathbf{p},r}(x)\Big] \\ &= \mathrm{e}^{\mathrm{i}\eta}\sum_{\mathbf{p},r}\Big[c_{-\mathbf{p},r}\,u_{\mathbf{p},r}(x)-d^{\dagger}_{-\mathbf{p},r}\,v_{\mathbf{p},r}(x)\Big] \end{aligned} \tag{4.163}$$

Hence, if we require

$$\mathcal{P}\,c_{\mathbf{p},r}\,\mathcal{P}^{\dagger} = \mathrm{e}^{\mathrm{i}\eta}\,c_{-\,\mathbf{p},r} \qquad \mathcal{P}\,d_{\mathbf{p},r}\,\mathcal{P}^{\dagger} = -\,\mathrm{e}^{-\,\mathrm{i}\eta}\,d_{-\,\mathbf{p},r} \tag{4.164}$$

then the transormation law (4.162) is fulfilled. Furthermore, it can be readily verified from the expressions (4.65) and (4.66) that we have

$$\mathcal{P}\, P^{\mu}\, \mathcal{P}^{\dagger} \;=\; P_{\mu} \qquad (4.165)$$

If we choose $\eta = 2\pi k \quad (k \in \mathbb{Z})$ then the relative parity of the particle-antiparticle system is equal to minus one and, contextually, the square of the parity operator (or space inversion operator) $\mathcal{P}$ is equal to the identity operator, that is

$$\mathcal{P}^{\,2} = \mathbf{I} \qquad \mathcal{P}^{\,\dagger} = \mathcal{P} = \mathcal{P}^{\,-1} \qquad (\eta = 2\pi k\,,\;\; k \in \mathbb{Z})$$

As a consequence we can write for instance

$$\mathcal{P}\,\psi\,(t,\mathbf{x})\,\mathcal{P} = \gamma^{0}\,\psi\,(t,-\,\mathbf{x}) \qquad \mathcal{P}\,\psi^{\,\dagger}\,(t,\mathbf{x})\,\mathcal{P} = \bar{\psi}\,(t,-\,\mathbf{x}) \qquad (4.166)$$

$$\mathcal{P}\,\bar{\psi}\,(t,\mathbf{x})\,\mathcal{P} = \mathcal{P}\,\psi^{\,\dagger}\,(t,\mathbf{x})\,\gamma^{0}\,\mathcal{P} = \mathcal{P}\,\psi^{\,\dagger}\,(t,\mathbf{x})\,\mathcal{P}\,\gamma^{0} = \bar{\psi}\,(t,-\,\mathbf{x})\,\gamma^{0}$$

$$\mathcal{P}\,(\bar{\psi}\,\psi)\,(t,\mathbf{x})\,\mathcal{P} = +\,(\bar{\psi}\,\psi)\,(t,-\,\mathbf{x}) \qquad (4.167)$$

$$\mathcal{P}\,(\bar{\psi}\,\gamma_{5}\,\psi)\,(t,\mathbf{x})\,\mathcal{P} = -\,(\bar{\psi}\,\gamma_{5}\,\psi)\,(t,-\,\mathbf{x}) \qquad \text{et cetera} \qquad (4.168)$$

4.7.3 The Time Reversal

Let us turn now to the implementation of time reversal transformation. It is known[5] that the time reversal transformation in quantum mechanics is achieved by means of *antilinear and antiunitary operators* $\mathcal{T} \;:\; \mathcal{H} \;\rightarrow\; \mathcal{H}\,$, where $\mathcal{H}$ is a Hilbert space, which satisfy

$$\mathcal{T}\,(a\,|\,\alpha\,\rangle + b\,|\,\beta\,\rangle) = a^{*}\,\mathcal{T}\,|\,\alpha\,\rangle + b^{*}\,\mathcal{T}\,|\,\beta\,\rangle \qquad a, b \in \mathbb{C} \quad \alpha, \beta \in \mathcal{H}$$

$$\langle\,\alpha\,|\,\mathcal{T}^{\,\dagger}\mathcal{T}\,\beta\,\rangle \;=\; \langle\,\mathcal{T}\,\alpha\,|\,\mathcal{T}\,\beta\,\rangle \;=\; \langle\,\beta\,|\,\alpha\,\rangle \qquad \forall\,\alpha, \beta \in \mathcal{H}$$

so that invariance under time reversal requires that

$$[\,\mathcal{T}\,,\,P_{0}\,] \;=\; [\,\mathcal{T}\,,\,H\,] \;=\; 0$$

where $P_0 = H$ denotes the infinitesimal generator of the time translations of the system. The physical significance of $\mathcal{T}$ as the time reversal operator requires that, while spatial relations must be unchanged, all momenta and angular momenta must be reversed. Hence, we shall postulate the conditions

$$\mathcal{T}\,\mathbf{P}\,\mathcal{T}^{\,\dagger} \;=\; -\,\mathbf{P} \qquad \{\mathcal{T}\,,\,\mathbf{P}\} = 0 \qquad (4.169)$$

$$\mathcal{T}\,L^{\rho\sigma}\,\mathcal{T}^{\,\dagger} \;=\; -\,L^{\rho\sigma} \qquad \{\mathcal{T}\,,\,L^{\rho\sigma}\} = 0 \qquad (4.170)$$

$$\mathcal{T}\,S^{\rho\sigma}\,\mathcal{T}^{\,\dagger} \;=\; -\,S^{\rho\sigma} \qquad \{\mathcal{T}\,,\,S^{\rho\sigma}\} = 0 \qquad (4.171)$$

[5]See E.P. Wigner, *Group Theory and Its Application to Quantum Mechanics of Atomic Spectra*, translated by J.J. Griffin, Academic Press, New York, 1959, Appendix to Chapter 20, p. 233. See also V. Bargmann, J. Math. Phys. **5** (1964) 862.

Since the antiunitary time reversal operator $\mathcal{T}$ is defined to reverse the sign of all momenta and spins we therefore require

$$\mathcal{T}\, c_{\mathbf{p}\,,r}\, \mathcal{T}^{\,\dagger} \;=\; \exp\{\,\mathrm{i}\,\eta_{\mathbf{p}\,,r}\}\, c_{-\mathbf{p}\,,-r} \tag{4.172}$$
$$\mathcal{T}\, d_{\mathbf{p}\,,r}\, \mathcal{T}^{\,\dagger} \;=\; \exp\{\,\mathrm{i}\,\zeta_{\mathbf{p}\,,r}\}\, d_{-\mathbf{p}\,,-r} \tag{4.173}$$
$$\mathcal{T}\, c^{\dagger}_{\mathbf{p}\,,r}\, \mathcal{T}^{\,\dagger} \;=\; \exp\{-\,\mathrm{i}\,\eta_{\mathbf{p}\,,r}\}\, c^{\dagger}_{-\mathbf{p}\,,-r} \tag{4.174}$$
$$\mathcal{T}\, d^{\dagger}_{\mathbf{p}\,,r}\, \mathcal{T}^{\,\dagger} \;=\; \exp\{-\,\mathrm{i}\,\zeta_{\mathbf{p}\,,r}\}\, d^{\dagger}_{-\mathbf{p}\,,-r} \tag{4.175}$$

where the notation $-\,r$ refers to the change of sign of the spin eigenvalue, that is

$$c_{-\mathbf{p}\,,-1} = c_{-\mathbf{p}\,,2} \qquad c_{-\mathbf{p}\,,-2} = c_{-\mathbf{p}\,,1} \quad \text{et cetera}$$

Although the phase factors $\exp\{\,\mathrm{i}\,\eta_{\mathbf{p},r}\}$ and $\exp\{\,\mathrm{i}\,\zeta_{\mathbf{p},r}\}$ are arbitrary, it is always possible to choose them in such a manner that the (Dirac spinor) fields undergo utmost simple transformation laws under time reversal.

This means that, in order to obtain the time reversed spinor field operator $\mathcal{T}\,\psi(x)\,\mathcal{T}^{\,\dagger}$, we have to perform on the complex conjugated c-number part (*i.e.* not operator part) of the spinor some unitary operation in such a way that eventually we come to the transformation rule

$$\psi^{\,\prime}(x^{\,\prime}) \;=\; \psi^{\,\prime}(-\,t,\mathbf{x}) = \mathcal{T}\,\psi\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger} \tag{4.176}$$

By inserting once again the the normal mode expansion (4.28) we get, up to some trivial substitutions in the integrands,

$$\begin{aligned}
\mathcal{T}\,\psi\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger} \;=\; & \sum_{\mathbf{p}\,,r} c_{\mathbf{p}\,,r}\, \exp\{\,\mathrm{i}\,\eta_{-\mathbf{p}\,,-r}\}\, u^{*}_{-r}\,(-\mathbf{p}) \\
\times\; & [\,(2\pi)^{3}\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, \exp\{-\,\mathrm{i}\,(-t)\,\omega_{\mathbf{p}} + \mathrm{i}\,\mathbf{p}\cdot\mathbf{x}\,\} \\
+\; & \sum_{\mathbf{p}\,,r} d^{\dagger}_{\mathbf{p}\,,r}\, \exp\{-\,\mathrm{i}\,\zeta_{-\mathbf{p}\,,-r}\}\, v^{*}_{-r}\,(-\mathbf{p}) \\
\times\; & [\,(2\pi)^{3}\, 2\omega_{\mathbf{p}}\,]^{-1/2}\, \exp\{\mathrm{i}\,(-t)\,\omega_{\mathbf{p}} - \mathrm{i}\,\mathbf{p}\cdot\mathbf{x}\,\}
\end{aligned}$$

In arriving at this equation, the antiunitarity nature of $\mathcal{T}$ has been exploited and has resulted in complex conjugation. It is easily seen that the right-hand side of this equation becomes a local expression for the field at time $-\,t$ if a 4×4 matrix Θ can be found such that

$$u^{*}_{-r}\,(-\mathbf{p})\, \exp\{\,\mathrm{i}\,\eta_{-\mathbf{p}\,,-r}\} \;=\; \Theta\, u_{r}\,(\mathbf{p}) \tag{4.177}$$
$$v^{*}_{-r}\,(-\mathbf{p})\, \exp\{-\,\mathrm{i}\,\zeta_{-\mathbf{p}\,,-r}\} \;=\; \Theta\, v_{r}\,(\mathbf{p}) \tag{4.178}$$

The closure relation (4.19) and the orthonormality relation (4.18) imply that Θ must be unitary. Furthermore, the above relations (4.177) and (4.178) are consistent with the spin-states eigenvalue equations (4.17) only if Θ satisfies the conditions

$$\begin{gathered}[\,\Theta\,,\,H\,] \;=\; 0 \\ \left(\alpha^{\,k}\right)^{*}\,\Theta = -\,\Theta\,\alpha^{\,k} \qquad (k=1,2,3) \qquad \beta^{*}\,\Theta = \Theta\,\beta\end{gathered}$$

As a matter of fact we have, for example,

$$\begin{aligned} H\,u_r\,(\mathbf{p}) = \left(\alpha^{\,k}\,p^k + \beta\,M\right)\,u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,u_r\,(\mathbf{p}) \\ \left(-\,\alpha^{\,k\,*}\,p^k + \beta^*\,M\right)\,u_r^*\,(-\,\mathbf{p}) &= \omega_{\mathbf{p}}\,u_r^*\,(-\,\mathbf{p}) \\ \left(-\,\alpha^{\,k\,*}\,p^k + \beta^*\,M\right)\,\Theta\,u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,\Theta\,u_r\,(\mathbf{p}) \\ \Theta\left(\alpha^{\,k}\,p^k + \beta\,M\right)\,u_r\,(\mathbf{p}) = \Theta\,H\,u_r\,(\mathbf{p}) &= \omega_{\mathbf{p}}\,\Theta\,u_r\,(\mathbf{p}) \end{aligned}$$

The unitary solution to these equations is unique except for an arbitrary phase factor that we can suitably choose to be equal to one, that means

$$\eta_{\,\mathbf{p}\,,\,r} = \zeta_{\,\mathbf{p}\,,\,r} = 2k\,\pi\,\mathrm{i} \qquad (\,k\in\mathbb{Z}\,,\ \forall\,\mathbf{p}\in\mathbb{R}^3\,,\ r=1,2\,)$$

In the spinorial-chiral-Weyl representation (2.66), the matrix

$$\Theta \;=\; -\,\gamma^1\,\gamma^3 \;=\; -\,\mathrm{i}\,\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}$$

is a solution. It has the important property

$$\Theta\,\Theta^* \;=\; -\,\mathbb{I} \qquad\qquad \Theta\,\Theta^\dagger \;=\; \mathbb{I}$$

which can be proved to hold true independently of the representation of the Dirac matrices. It follows that the Dirac particle-antiparticle spinor quantum wave field is transformed under time reversal according to

$$\begin{aligned} \mathcal{T}\,\psi\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &\;=\; \Theta\,\psi\,(-\,t,\mathbf{x}) \\ \mathcal{T}\,\psi^{\,\dagger}\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &\;=\; \psi^{\,\dagger}\,(-\,t,\mathbf{x})\,\Theta^{\,\dagger} \end{aligned} \tag{4.179}$$

It is now rather easy to derive the action of the time reversal operator on various bilinears. First of all we have

$$\begin{aligned} \mathcal{T}\,\bar\psi\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger} &\;=\; \mathcal{T}\,\psi^{\,\dagger}\,(t,\mathbf{x})\,\mathcal{T}^{\,\dagger}\,\gamma^{0\,*} \\ &\;=\; \psi^{\,\dagger}\,(-\,t,\mathbf{x})\,\Theta^{\,\dagger}\,\gamma^{0\,*} \\ &\;=\; \bar\psi\,(-\,t,\mathbf{x})\,\gamma^1\,\gamma^3 \end{aligned} \tag{4.180}$$

Then the transformation law for the scalar mass bilinear under time reversal becomes

$$\begin{aligned} \mathcal{T}\,\bar{\psi}\,(t,\mathbf{x})\,\psi\,(t,\mathbf{x})\,\mathcal{T}^{\dagger} &= \bar{\psi}\,(-\,t,\mathbf{x})\,\gamma^{1}\,\gamma^{3}\left(-\,\gamma^{1}\,\gamma^{3}\right)\,\psi\,(-\,t,\mathbf{x}) \\ &= +\,\bar{\psi}\,(-\,t,\mathbf{x})\,\psi\,(-\,t,\mathbf{x}) \end{aligned} \tag{4.181}$$

A quite analogous calculation gives for instance

$$\mathcal{T}\,\bar{\psi}\,(t,\mathbf{x})\,\gamma^{\mu}\,\psi\,(t,\mathbf{x})\,\mathcal{T}^{\dagger} \;=\; \begin{cases} \psi^{\dagger}\,(-\,t,\mathbf{x})\,\psi\,(-\,t,\mathbf{x}) & \text{for } \mu=0 \\ -\,\bar{\psi}\,(-\,t,\mathbf{x})\gamma^{k}\,\psi\,(-\,t,\mathbf{x}) & \text{for } \mu=1,2,3 \end{cases}$$

References

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4.8 Problems

1. **The Belifante tensor.** In general, for a relativistic wave field with non zero spin, the canonical energy momentum tensor needs not to be symmetric. Show that one can always find a term $\mathcal{B}^{\lambda\mu\nu}$ antisymmetric under $\lambda \to \mu$ or ν such that the Belifante tensor

$$\Theta^{\mu\nu}(x) = T^{\mu\nu}(x) - \partial_\lambda \mathcal{B}^{\lambda\mu\nu}(x)$$

is symmetric and the corresponding conserved Noether current for the Lorentz transformations is written in the form

$$M^{\mu\lambda\kappa}(x) = x^\lambda\,\Theta^{\mu\kappa}(x) - x^\kappa\,\Theta^{\mu\lambda}(x)$$

Solution. One can always decompose the canonical energy momentum tensor of the Dirac wave field into its symmetric and antisymmetric parts

$$\begin{aligned} T^{\mu\nu}(x) &= \frac{i}{4}\,\bar\psi(x)\,\gamma^\mu\,\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) + \frac{i}{4}\,\bar\psi(x)\,\gamma^\nu\,\overleftrightarrow{\partial}^{\,\mu}\,\psi(x) \\ &+ \frac{i}{4}\,\bar\psi(x)\,\gamma^\mu\,\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) - \frac{i}{4}\,\bar\psi(x)\,\gamma^\nu\,\overleftrightarrow{\partial}^{\,\mu}\,\psi(x) \end{aligned}$$

in which

$$\begin{aligned} T^{\mu\nu}(x) - T^{\nu\mu}(x) &= \frac{i}{2}\,\bar\psi(x)\,\gamma^\mu\,\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) - \frac{i}{2}\,\bar\psi(x)\,\gamma^\nu\,\overleftrightarrow{\partial}^{\,\mu}\,\psi(x) \\ &= \partial_\lambda\,S^{\lambda\mu\nu}(x) \end{aligned}$$

in accordance with eq. (4.44), where the third rank tensor of the spin angular momentum density of the Dirac field is defined by the Noether theorem and reads

$$S^{\lambda\mu\nu}(x) \equiv \frac{1}{2}\,\bar\psi(x)\,\{\gamma^\lambda\,,\,\sigma^{\mu\nu}\}\,\psi(x)$$

which enjoys by construction

$$S^{\lambda\mu\nu}(x) + S^{\lambda\nu\mu}(x) = 0 \qquad\qquad S^{\lambda\mu\nu}(x) = S^{\mu\nu\lambda}(x)$$

If we introduce the auxiliary quantity

$$\mathcal{B}^{\lambda\mu\nu}(x) \equiv \frac{1}{2}\Big(S^{\lambda\mu\nu}(x) - S^{\mu\lambda\nu}(x) - S^{\nu\lambda\mu}(x)\Big)$$

which is evidently related to a non-vanishing spin angular momentum density tensor and which fulfills by definition

$$\mathcal{B}^{\lambda\mu\nu} + \mathcal{B}^{\mu\lambda\nu} = 0 \qquad\qquad \partial_\lambda\,\mathcal{B}^{\lambda\mu\nu}(x) = \frac{1}{2}\,\partial_\lambda\,S^{\lambda\mu\nu}(x)$$

In fact we have

$$\begin{aligned}
2\partial_\lambda\,\mathcal{B}^{\lambda\mu\nu}(x) &= \partial_\lambda\,S^{\lambda\mu\nu}(x) - \partial_\lambda\,S^{\mu\lambda\nu}(x) - \partial_\lambda\,S^{\nu\lambda\mu}(x) \\
&= \partial_\lambda\,S^{\lambda\mu\nu}(x) + \partial_\lambda\,S^{\mu\nu\lambda}(x) + \partial_\lambda\,S^{\nu\mu\lambda}(x) \\
&= \partial_\lambda\,S^{\lambda\mu\nu}(x) + \partial_\lambda\,S^{\lambda\mu\nu}(x) + \partial_\lambda\,S^{\lambda\nu\mu}(x) \\
&= 2\partial_\lambda\,S^{\lambda\mu\nu}(x) + \partial_\lambda\,S^{\lambda\nu\mu}(x) = \partial_\lambda\,S^{\lambda\mu\nu}(x)
\end{aligned}$$

Then the symmetric energy momentum tensor is obtained by setting

$$\begin{aligned}
\Theta^{\mu\nu}(x) &\stackrel{\mathrm{def}}{=} \frac{1}{2}\Big(T^{\mu\nu}(x) + T^{\nu\mu}(x)\Big) \\
&= \frac{i}{4}\,\bar\psi(x)\,\gamma^\mu\,\overleftrightarrow{\partial}^{\,\nu}\,\psi(x) + \frac{i}{4}\,\bar\psi(x)\,\gamma^\nu\,\overleftrightarrow{\partial}^{\,\mu}\,\psi(x) \\
&= T^{\mu\nu}(x) - \frac{1}{2}\,\partial_\lambda\,S^{\lambda\mu\nu}(x) \\
&= T^{\mu\nu}(x) - \partial_\lambda\,\mathcal{B}^{\lambda\mu\nu}(x)
\end{aligned}$$

which apparently satisfies the continuity equation since

$$\partial_\mu\Theta^{\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) - \partial_\lambda\partial_\mu\,\mathcal{B}^{\lambda\mu\nu}(x) = \partial_\mu T^{\mu\nu}(x) = 0$$

Consequently the total angular momentum density tensor for the Dirac field can be written in the *purely orbital* form. In fact we have

$$\begin{aligned}
M^{\mu\kappa\lambda}(x) &= x^\kappa\,\Theta^{\mu\lambda}(x) - x^\lambda\,\Theta^{\mu\kappa}(x) \\
&= x^\kappa\,T^{\mu\lambda}(x) + \frac{1}{2}\,x^\kappa\,\partial_\rho\,S^{\rho\mu\lambda}(x) \\
&- x^\lambda\,T^{\mu\kappa}(x) - \frac{1}{2}\,x^\lambda\,\partial_\rho\,S^{\rho\mu\kappa}(x) \\
&= x^\kappa\,T^{\mu\lambda}(x) - x^\lambda\,T^{\mu\kappa}(x) \\
&- \frac{1}{2}\Big[\,S^{\kappa\mu\lambda}(x) - S^{\lambda\mu\kappa}(x)\Big] \\
&- \frac{1}{2}\,\partial_\rho\Big[\,x^\lambda\,S^{\rho\mu\kappa}(x) - x^\kappa\,S^{\rho\mu\lambda}(x)\Big]
\end{aligned}$$

and taking the symmetry properties of the spin angular momentum density tensor suitably into account

$$\begin{aligned} M^{\mu\kappa\lambda}(x) &= x^{\kappa}\,\Theta^{\mu\lambda}(x) - x^{\lambda}\,\Theta^{\mu\kappa}(x) \\ &= x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x) \\ &+ \frac{1}{2}\,\partial_{\rho}\left[\,x^{\kappa}\,S^{\lambda\rho\mu}(x) - x^{\lambda}\,S^{\kappa\rho\mu}(x)\,\right] \end{aligned}$$

in such a manner that the continuity equations hold true : namely,

$$\begin{aligned} \partial_{\mu} M^{\mu\kappa\lambda}(x) &= \partial_{\mu}\left(x^{\kappa}\,\Theta^{\mu\lambda}(x) - x^{\lambda}\,\Theta^{\mu\kappa}(x)\right) \\ &= \partial_{\mu}\Big(x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x)\Big) \\ &- \frac{1}{2}\,\partial_{\rho}\,\partial_{\mu}\Big(x^{\lambda}\,S^{\kappa\rho\mu}(x) - x^{\kappa}\,S^{\lambda\rho\mu}(x)\Big) \\ &= \partial_{\mu}\Big(x^{\kappa}\,T^{\mu\lambda}(x) - x^{\lambda}\,T^{\mu\kappa}(x) + S^{\mu\lambda\kappa}(x)\Big) = 0 \end{aligned}$$

2. **Majorana fermions**. One can write a relativistically invariant field equation for a massive 2-component left Weyl spinor wave field ψ_L that transforms according to (2.41). Call such a 2-component field $\chi_a(x)$ $(a = 1, 2)$. Let us consider the Weyl spinor wave field as a classical anticommuting field, *i.e.* a *Grassmann valued* left Weyl spinor field function over the Minkowski spacetime which satisfies

$$\{\,\chi_a(x)\,,\,\chi_b(y)\,\} = 0 \qquad (\,x, y \in \mathcal{M} \qquad a, b = 1, 2\,)$$

together with the complex conjugation rule

$$(\chi_1\chi_2)^* = \chi_2^*\chi_1^* = -\,\chi_1^*\chi_2^*$$

so to imitate the hermitean conjugation of quantum fields.

(a) Show that the classical lagrangian

$$\mathcal{L}_L = \frac{1}{2}\,\chi^{\dagger}(x)\,\sigma^{\mu}\,i\,\overleftrightarrow{\partial}_{\mu}\,\chi(x) + \frac{m}{2}\,\left[\,\chi^{\top}(x)\,\sigma_2\,\chi(x) + \chi^{\dagger}(x)\,\sigma_2\,\chi^*(x)\,\right]$$

is real with $\chi^{\dagger} = (\chi^*)^{\top}$ and yields the Majorana wave field equation

$$i\,\sigma^{\mu}\,\partial_{\mu}\,\chi(x) + m\,\sigma_2\,\chi^*(x) = 0$$

where $\sigma^{\mu} = (\mathbf{1}\,,\,-\sigma_k)$. That is, show that this equation, named the *Majorana field equation*, is relativistically invariant and that it implies the Klein-Gordon equation $(\Box + m^2)\,\chi(x) = 0$. This form of the fermion mass is called a *Majorana mass term*.

(b) Show that all we have obtained before can be formulated in terms of a self-conjugated bispinor called the free Majorana spinor ψ_M and, in particular, derive the symmetries of the corresponding action.

(c) Perform the quantum theory of the Majorana massive field.

Solution

(a) Consider the transformation rule (2.55)

$$\Lambda_L^{\dagger}\,\sigma^{\mu}\,\Lambda_L = \Lambda^{\mu}_{\ \nu}\,\sigma^{\nu}$$

Then we immediately get

$$\begin{aligned}
[\,\chi'(x')\,]^{\dagger}\,\sigma^{\mu}\,i\,\overleftrightarrow{\partial}'_{\!\mu}\,\chi'(x') &= \chi^{\dagger}(x)\,\Lambda_L^{\dagger}\,\sigma^{\mu}\,\Lambda_L\,i\,\overleftrightarrow{\partial}_{\!\rho}\,\chi(x)\,\Lambda_{\mu}^{\ \rho} \\
&= \chi^{\dagger}(x)\,\Lambda^{\mu}_{\ \nu}\,\sigma^{\nu}\,i\,\overleftrightarrow{\partial}_{\!\rho}\,\chi(x)\,\Lambda_{\mu}^{\ \rho} \\
&= \chi^{\dagger}(x)\,\sigma^{\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\,\chi(x)
\end{aligned}$$

which vindicates once more the Poincaré invariance of the left Weyl kinetic lagrangian

$$\mathfrak{T}_L\,[\,\chi\,] = \frac{1}{2}\,\chi^{\dagger}(x)\,\sigma^{\mu}\,i\,\overleftrightarrow{\partial}_{\!\mu}\,\chi(x)$$

Furthermore we have

$$\begin{aligned}
\chi'^{\top}(x')\,\sigma_2\,\chi'(x') &= \chi^{\top}(x)\,\Lambda_L^{\top}\,\sigma_2\,\Lambda_L\,\chi(x) \\
&= \chi^{\top}(x)\,\sigma^2\,\Lambda_L^{-1}\,(\sigma_2)^2\,\Lambda_L\,\chi(x) \\
&= \chi^{\top}(x)\,\sigma_2\,\chi(x) \\
&= -\,i\,\chi_1(x)\,\chi_2(x) + \,i\,\chi_2(x)\,\chi_1(x) \\
&= -\,2\,i\,\chi_1(x)\,\chi_2(x)
\end{aligned}$$

for anticommuting Grassmann valued Weyl spinor fields. It follows therefrom that the mass term is the real Lorentz scalar

$$\begin{aligned}
\mathcal{L}_L^{m} &= -\,i\,m\left[\,\chi_1(x)\,\chi_2(x) + \chi_1^{*}(x)\,\chi_2^{*}(x)\,\right] \\
&= i\,m\left[\,\chi_2^{*}(x)\,\chi_1^{*}(x) + \chi_2(x)\,\chi_1(x)\,\right] \;=\; (\mathcal{L}_L^{m})^{*}
\end{aligned}$$

The Lagrange density can be rewritten as

$$
\begin{aligned}
\mathcal{L}_L &= \frac{1}{2}\,\chi^\dagger(x)\,\sigma^\mu\, i\overleftrightarrow{\partial}_\mu\chi(x) + \frac{m}{2}\,\left[\,\chi^\top(x)\,\sigma_2\,\chi(x) + \chi^\dagger(x)\,\sigma_2\,\chi^*(x)\,\right] \\
&= \chi^\dagger(x)\,\sigma^\mu\, i\,\partial_\mu\chi(x) + \frac{m}{2}\,\left[\,\chi^\top(x)\,\sigma_2\,\chi(x) + \chi^\dagger(x)\,\sigma_2\,\chi^*(x)\,\right] \\
&- \frac{i}{2}\,\partial_\mu\Big(\chi^\dagger(x)\,\sigma^\mu\,\chi(x)\Big) \\
&\doteq \chi^\dagger(x)\,\sigma^\mu\, i\,\partial_\mu\chi(x) + \frac{m}{2}\,\left[\,\chi^\top(x)\,\sigma_2\,\chi(x) + \chi^\dagger(x)\,\sigma_2\,\chi^*(x)\,\right]
\end{aligned}
$$

where $\doteq$ means that the 4-divergence term can be disregarded as it does not contribute to the equations of motion. If we treat $\chi_a(x)$ $(a = 1, 2)$ and $\chi_a^*(x)$ $(a = 1, 2)$ as independent field variables, then the Euler-Lagrange field equations yield

$$\partial_\mu\,\delta\,\mathcal{L}_L\,/\,\partial_\mu\,\delta\,\chi(x) = i\,\partial_\mu\chi^\dagger(x)\,\sigma^\mu = \delta\,\mathcal{L}_L\,/\,\delta\,\chi(x) = m\,\chi^\top(x)\,\sigma_2$$

hence, taking the transposed and complex conjugate equation

$$i\,\sigma^\mu\,\partial_\mu\chi(x) \,+\, m\,\sigma_2\,\chi^*(x) \;=\; 0 \qquad\qquad (I)$$

which is the Majorana field equation. Multiplication to the left by σ_2 and taking complex conjugation gives

$$i\sigma_2\,\partial_0\,\chi^*(x) -\, i\sigma_2\,\sigma_k^*\,\partial_k\,\chi^*(x) + m\chi(x) = 0$$

Remembering that $\sigma_2\,\sigma_k^*\,\sigma_2 = -\,\sigma_k$ and that $\bar\sigma^\mu = (\,\mathbf{1}\,,\,\sigma_k\,)$ we come to the equivalent form of the Majorana left equation, *i.e.*

$$i\,\bar\sigma^\mu\,\partial_\mu\chi^*(x) + m\,\sigma_2\,\chi(x) = 0 \qquad\qquad (II)$$

Now, if act from the left with the operator $i\,\bar\sigma^\nu\,\partial_\nu$ to equation $(\,I\,)$ and use equation $(\,II\,)$ we obtain

$$\bar\sigma^\nu\sigma^\mu\,\partial_\nu\,\partial_\mu\chi(x) + m^2\,\chi(x) = (\,\Box + m^2\,)\,\chi(x) = 0$$

so that the left-handed Weyl massive spinor satisfies the Klein-Gordon wave equation.

(b) According to (2.82) we can introduce the Majorana left-handed self-conjugated bispinor

$$\chi_M(x) = \begin{pmatrix} \chi(x) \\ -\,\sigma_2\,\chi^*(x) \end{pmatrix} = \chi_M^c(x) \qquad\qquad (III)$$

with the Lagrange density

$$\mathcal{L}_M = \frac{1}{4}\,\bar{\chi}_M\,(x)\;\mathrm{i}\overleftrightarrow{\partial\!\!\!/}\,\chi_M\,(x)\;\;-\;\;\frac{m}{2}\,\bar{\chi}_M\,(x)\,\chi_M\,(x)$$

in such a manner that the Majorana mass term can be written in the two equivalent forms

$$\mathcal{L}_L^m \;=\; \frac{m}{2}\,\left[\,\chi^\top(x)\,\sigma_2\,\chi(x) + \chi^\dagger(x)\,\sigma_2\,\chi^*(x)\,\right] \;=\; -\,\frac{m}{2}\,\bar{\chi}_M\,(x)\,\chi_M\,(x)$$

so that it is clear that the Majorana's action $\int \mathrm{d}x\,\mathcal{L}_M$ is no longer invariant under the phase transformation

$$\chi\,(x)\;\longmapsto\;\chi'\,(x) = \chi\,(x)\,e^{\,i\alpha}$$

Taking the Grassmann valued Majorana spinor wave field $\chi_M\,(x)$ to be defined by the self-conjugation constraint $(\,III\,)$, it is easy to see that the pair of coupled Weyl equations $(\,I\,)$ and $(\,II\,)$ is equivalent to the single bispinor equation

$$(\alpha^\mu\,i\,\partial_\mu - \beta\,m)\;\chi_M\,(x) = 0 \qquad\qquad (\,IV\,)$$

in which

$$\alpha^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{pmatrix} \qquad \beta = \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^\mu = \beta\,\alpha^\mu$$

while the Majorana lagrangian then becomes

$$\mathcal{L}_M = \frac{1}{4}\,\chi_M^\dagger\,(x)\,\alpha^\mu\,i\,\overleftrightarrow{\partial}_\mu\chi_M\,(x) - \frac{m}{2}\,\chi_M^\dagger\,(x)\,\beta\,\chi_M\,(x)$$

It is immediate to verify that the bispinor form $(\,IV\,)$ of the field equations does coincide with the two equivalent forms $(\,I\,)$ and $(\,II\,)$ of the Majorana wave field equation : namely,

$$\begin{cases} i\,\sigma^\mu\,\partial_\mu\,\chi\,(x) + m\,\sigma_2\,\chi^*(x) &= \;0 \\ i\,\bar{\sigma}^\mu\,\sigma_2\,\partial_\mu\,\chi^*\,(x) + m\,\chi\,(x) &= \;0 \end{cases}$$

Since the Majorana spinor wave field $\chi_M\,(x)$ has the constraint $(\,III\,)$ which relates the lower two components to the complex conjugate of the upper two components, a representation must exist which makes the Majorana spinor wave field real, with the previous two independent complex variables $\chi_a \in \mathbb{C}$ $(a = 1, 2)$ replaced by the four real variables

$\psi_{M,\alpha} \in \mathbb{R}\ (\alpha = 1L, 2L, 1R, 2R)$. To obtain this real representation, we note that

$$\chi_M = \left(\begin{array}{c} \chi \\ -\sigma_2\,\chi^* \end{array} \right) \qquad \chi_M^* = \left(\begin{array}{cc} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{array} \right) \chi_M$$

A transformation to real bispinor fields $\psi_M = \psi_M^*$ can be made by writing

$$\chi_M = S\,\psi_M$$

whence

$$S^* = \left(\begin{array}{cc} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{array} \right) S$$

Now, if we set

$$\left(\begin{array}{cc} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{array} \right) = i\gamma^2 \equiv \rho_2$$

$$\rho_2 = \rho_2^\dagger \qquad \rho_2^2 = \mathbb{I}$$

so that

$$\exp\{i\,\rho_2\,\theta\} = \mathbb{I}\,\cos\theta + i\,\rho_2\,\sin\theta$$

then the solution for the above relation is the unitary matrix

$$S = \exp\{-\pi\,i\,\rho_2\,/\,4\} = \frac{\sqrt{2}}{2}\left(\mathbb{I} - i\,\rho_2\right)$$

which fulfills

$$S = \frac{\sqrt{2}}{2} \left(\begin{array}{cc} \mathbf{1} & \sigma_2 \\ -\sigma_2 & \mathbf{1} \end{array} \right) = \frac{\sqrt{2}}{2}\left(\mathbb{I} + \gamma^2\right)$$

or even more explicitly

$$S = \frac{\sqrt{2}}{2} \left(\begin{array}{cccc} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{array} \right)$$

It follows that we can suitably make use of the so called *Majorana representation for the gamma matrices* which is given by the similarity and unitary transformation acting on the gamma matrices in the Weyl representation, *viz.,*

$$\gamma_M^\mu \equiv S^\dagger\,\gamma^\mu\,S$$

$$\gamma_M^0 = \begin{pmatrix} -\,\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ -\,i & 0 & 0 & 0 \\ 0 & 0 & 0 & -\,i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_M^1 = \begin{pmatrix} -\,i\,\sigma_3 & 0 \\ 0 & -\,i\,\sigma_3 \end{pmatrix} = \begin{pmatrix} -\,i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -\,i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$\gamma_M^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\,\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\,i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -\,i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_M^3 = \begin{pmatrix} i\,\sigma_1 & 0 \\ 0 & i\,\sigma_1 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_M^5 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\,i \\ 0 & 0 & i & 1 \\ 0 & -\,i & 0 & 0 \\ i & 0 & i & 0 \end{pmatrix}$$

which satisfy by direct inspection

$$\{\gamma_M^\mu\,,\,\gamma_M^\nu\} = 2\,g^{\mu\nu} \qquad \{\gamma_M^\nu\,,\,\gamma_M^5\} = 0$$

$$\gamma_M^0 = \gamma_M^{0\,\dagger} \qquad \gamma_M^k = -\,\gamma_M^{k\,\dagger} \qquad \gamma_M^5 = \gamma_M^{5\,\dagger}$$

$$\gamma_M^\nu \;=\; -\,\gamma_M^{\nu\,*} \qquad \gamma_M^5 \;=\; -\,\gamma_M^{5\,*}$$

Then the Majorana lagrangian and the ensuing Majorana wave field equation take the form

$$\mathcal{L}_M = \frac{1}{4}\,\psi_M^\top(x)\,\alpha_M^\nu\,i\,\overleftrightarrow{\partial}_\nu\,\psi_M(x) - \frac{m}{2}\,\psi_M^\top(x)\,\beta_M\,\psi_M(x)$$

$$(\mathrm{i}\,\partial\!\!\!/_M - m)\,\psi_M(x) = 0 \qquad \psi_M(x) = \psi_M^*(x)$$

$$\alpha_M^\nu = \gamma_M^0\,\gamma_M^\nu \qquad \alpha_M^0 = \mathbb{I} \qquad \beta_M \equiv \gamma_M^0$$

The only relic internal symmetry of the Majorana action is the discrete $\mathbb{Z}_2$ symmetry, *i.e.* $\psi_M(x) \longmapsto -\,\psi_M(x)$. The Majorana hamiltonian reads

$$H_M = \alpha_M^k\,\hat{p}^{\,k} + m\,\beta_M \qquad (\,\hat{p}^{\,k} = -\,i\,\nabla_k\,)$$

To solve the Majorana wave equation we set

$$\psi_M(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^{3/2}}\ \widetilde{\psi}_M(p)\ \exp\{-\,i\,p\cdot x\}$$

with the reality condition

$$\widetilde{\psi}^*_M(p) = \widetilde{\psi}_M(-\,p)$$

so that

$$(\not{p}_M - m)\,\widetilde{\psi}_M(p) = 0 \qquad \not{p}_M \equiv p_\nu\,\gamma^\nu_M$$

which implies

$$\begin{aligned} &\widetilde{\psi}_{M,\alpha}(p) = (\not{p}_M + m)_{\alpha\beta}\,\widetilde{\phi}_\beta(p) \\ &\left(p^2 - m^2\right)\,\widetilde{\phi}_\alpha(p) = 0 \qquad\qquad (\alpha = 1,2,3,4) \\ &\widetilde{\phi}_\alpha(p) = \delta\left(p^2 - m^2\right)\ f_\alpha(p) \end{aligned}$$

Furthermore, from the reality condition on the Majorana spinor field

$$\widetilde{\psi}^*_{M,\alpha}(p) = \widetilde{\psi}_{M,\alpha}(-\,p)$$

we find

$$\begin{aligned} \widetilde{\psi}^*_{M,\alpha}(p) &= (\not{p}^*_M + m)_{\alpha\beta}\,\widetilde{\phi}^*_\beta(p) \\ &= (-\not{p}_M + m)_{\alpha\beta}\,\widetilde{\phi}^*_\beta(p) \\ &= \widetilde{\psi}_{M,\alpha}(-\,p) \qquad \Longleftrightarrow \qquad f^*_\beta(p) = f_\beta(-\,p) \end{aligned}$$

thanks to the circumstance that the gamma matrices in the Majorana representation are purely imaginary. Then we can write

$$\begin{aligned} \psi_M(x) &= \int_{-\infty}^{\infty}\mathrm{d}p_0 \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\right]^{-1/2}\ \theta(p_0) \\ &\times\ (\not{p}_M + m)_{\alpha\beta}\,f_\beta(p)\,\delta(p_0 - \omega_{\mathbf{p}})\ \exp\{-\,i\,p\cdot x\} \\ &+\ \int_{-\infty}^{\infty}\mathrm{d}p_0 \int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\right]^{-1/2}\ \theta(-\,p_0) \\ &\times\ (\not{p}_M + m)_{\alpha\beta}\,f_\beta(p)\,\delta(p_0 + \omega_{\mathbf{p}})\ \exp\{-\,i\,p\cdot x\} \\ &= 2m\int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\right]^{-1/2}\mathcal{E}^+_M(p)\,f(p)\,e^{-\,ipx} \\ &+\ 2m\int \mathrm{d}\mathbf{p}\ \left[\,(2\pi)^3\,2\omega_{\mathbf{p}}\,\right]^{-1/2}\mathcal{E}^-_M(p)\,f^*(p)\,e^{\,ipx} \\ &\stackrel{\mathrm{def}}{=} (2\pi)^{-3/2}\sum_{\mathbf{p}}\ \left[\,\mathcal{E}^+_M(p)\ a_{\mathbf{p}}\,e^{-\,ipx} + \mathcal{E}^-_M(p)\ a^*_{\mathbf{p}}\,e^{\,ipx}\,\right] \\ &= \psi^*_M(x) \end{aligned}$$

where $p_0 = \omega_{\mathbf{p}}$, whereas $a_{\mathbf{p}} = 2m\, f(p)/\sqrt{2\omega_{\mathbf{p}}}$. The projectors onto the spin states are

$$\mathcal{E}^{\pm}_{M}(p) = (m \pm \not{p}_M)\,/2m \qquad\qquad (p_0 = \omega_{\mathbf{p}})$$

with

$$[\,\mathcal{E}^{\pm}_{M}(p)\,]^{*} = \mathcal{E}^{\mp}_{M}(p) \qquad [\,\mathcal{E}^{\pm}_{M}(p)\,]^{\dagger} = \mathcal{E}^{\pm}_{M}(\tilde{p}) \qquad (\,\tilde{p}^{\mu} = p_{\mu}\,)$$

$$[\,\mathcal{E}^{\pm}_{M}(p)\,]^{2} = \mathcal{E}^{\pm}_{M}(p) \qquad \mathcal{E}^{\pm}_{M}(p)\,\mathcal{E}^{\mp}_{M}(p) = 0$$

$$\operatorname{tr}\mathcal{E}^{\pm}_{M}(p) = 2 \qquad \mathcal{E}^{+}_{M}(p) + \mathcal{E}^{-}_{M}(p) = \mathbb{I}$$

Now, in order to set up the spin states of the left Majorana field, let me start from the real eigenvectors of the matrix γ^0 that are

$$\xi_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

which do indeed satisfy by direct inspection

$$\gamma^0\,\xi_r = \xi_r \qquad \xi_r^{\top}\,\gamma^k\,\xi_s = 0 \qquad \forall\, r,s = 1,2 \,\vee\, k = 1,2,3$$

and define

$$\tilde{\xi}_r \equiv S^{\dagger}\,\xi_r = \frac{\sqrt{2}}{2}\left(\mathbb{I} - \gamma^2\right)\xi_r \qquad (\,r = 1,2\,)$$

$$\tilde{\xi}^{*}_r \equiv S\,\xi_r = \frac{\sqrt{2}}{2}\left(\mathbb{I} + \gamma^2\right)\xi_r \qquad (\,r = 1,2\,)$$

$$\tilde{\xi}^{\dagger}_r \equiv \xi_r^{\top}\,S = \xi_r^{\top}\,\frac{\sqrt{2}}{2}\left(\mathbb{I} + \gamma^2\right) \qquad (\,r = 1,2\,)$$

in such a manner that we have by construction

$$\gamma^0_M\,\tilde{\xi}_r = \tilde{\xi}_r \qquad \tilde{\xi}^{\dagger}_r\,\gamma^k_M\,\tilde{\xi}_s = 0 \qquad \forall\, r,s = 1,2 \,\vee\, k = 1,2,3$$

$$\tilde{\xi}^{\dagger}_r\,\tilde{\xi}_s = \xi^{\dagger}_r\,\xi_s = 2\,\delta_{rs} \qquad (\,r,s = 1,2\,)$$

Notice that the spin states $\tilde{\xi}_r$ $(\,r = 1,2\,)$ are in fact the two degenerate eigenstates of the Majorana hamiltonian in the massive neutral spinor particle rest frame $\mathbf{p} = 0$ with positive eigenvalue $p_0 = m$.

Then we define the Majorana left spin states as

$$\begin{cases} \tilde{u}_r\,(p) \equiv 2m\,(2\omega_{\mathbf{p}} + 2m)^{-\frac{1}{2}}\,\mathcal{E}_M^+\,(p)\,\tilde{\xi}_r \\ \tilde{u}_r^*\,(p) \equiv 2m\,(2\omega_{\mathbf{p}} + 2m)^{-\frac{1}{2}}\,\mathcal{E}_M^-\,(p)\,\tilde{\xi}_r^* \end{cases} \qquad (\,r = 1, 2,\; p_0 = \omega_{\mathbf{p}}\,)$$

which are the two eigenstates of the positive energy projector

$$\mathcal{E}_M^+\,(p)\,\tilde{u}_r\,(p) = \tilde{u}_r\,(p) \qquad (\,r = 1, 2, \quad p_0 = \omega_{\mathbf{p}}\,)$$

and are normalized according to

$$\tilde{u}_r^\dagger\,(p)\,\tilde{u}_s\,(p) = 2\,\omega_{\mathbf{p}}\,\delta_{rs}$$

In fact we have for instance

$$\begin{aligned} \tilde{u}_r^\dagger(p)\,\tilde{u}_s(p) &= (2\omega_{\mathbf{p}} + 2m)^{-1}\;\tilde{\xi}_r^\dagger\,(m + \tilde{\not p}_M)\,(m + \not p_M)\,\tilde{\xi}_s \\ &= (2\omega_{\mathbf{p}} + 2m)^{-1}\;\tilde{\xi}_r^\dagger\left(2\omega_{\mathbf{p}}^2 + 2m\,\omega_{\mathbf{p}}\right)\tilde{\xi}_s \\ &= 2\,\omega_{\mathbf{p}}\,\tfrac{1}{2}\,\tilde{\xi}_r^\dagger\,\tilde{\xi}_s = 2\,\omega_{\mathbf{p}}\,\delta_{rs} \end{aligned}$$

in which I have made use of the property

$$\tilde{\xi}_r^\dagger\,\gamma_M^k\,\gamma_M^0\,\tilde{\xi}_s = \tilde{\xi}_r^\dagger\,\gamma_M^k\,\tilde{\xi}_s = 0 \qquad \forall\, r, s = 1, 2 \,\vee\, k = 1, 2, 3$$

In conclusion, the normal mode decomposition of the Majorana real spinor wave field becomes

$$\psi_M\,(x) = \sum_{\mathbf{p}\,,\,r}\,\left[\,a_{\,\mathbf{p}\,,\,r}\,u_{\,\mathbf{p}\,,\,r}(x) + a_{\,\mathbf{p}\,,\,r}^*\,u_{\,\mathbf{p}\,,\,r}^*(x)\,\right] = \psi_M^*\,(x)$$

with

$$u_{\,\mathbf{p}\,,\,r}\,(x) \equiv [\,(2\pi)^3\,2\,\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\;\tilde{u}_r(p)\;\exp\{-\,i\,\omega_{\mathbf{p}}\,t + i\,\mathbf{p}\cdot\mathbf{x}\}$$

$$\begin{aligned} \{a_{\,\mathbf{p}\,,\,r}\,,\,a_{\,\mathbf{q}\,,\,s}\} = \{a_{\,\mathbf{p}\,,\,r}\,,\,a_{\,\mathbf{q}\,,\,s}^*\} = \{a_{\,\mathbf{p}\,,\,r}^*\,,\,a_{\,\mathbf{q}\,,\,s}^*\} \;=\; 0 \\ \forall\,\mathbf{p}\,,\,\mathbf{q} \in \mathbb{R}^3 \qquad \forall\, r\,,\, s = 1, 2 \end{aligned}$$

(c) The transition to the quantum theory is performed as usual by the introduction of the creation annihilation operators $a_{\,\mathbf{p}\,,\,r}$ and $a_{\,\mathbf{p}\,,\,s}^\dagger$ which satisfy the canonical anticommutation relations

$$\{a_{\,\mathbf{p}\,,\,r}\,,\,a_{\,\mathbf{q}\,,\,s}\} = 0 = \{a_{\,\mathbf{p}\,,\,r}^\dagger\,,\,a_{\,\mathbf{q}\,,\,s}^\dagger\}$$

$$
\begin{aligned}
\{a_{\mathbf{p},r}\,,\,a^{\dagger}_{\mathbf{q},s}\} &= \delta_{rs}\,\delta\,(\mathbf{p}-\mathbf{q}) \\
\forall\,\mathbf{p}\,,\,\mathbf{q}\in\mathbb{R}^3 &\qquad \forall\,r\,,\,s=1,2
\end{aligned}
$$

so that

$$\psi_M\,(x) = \sum_{\mathbf{p}\,,\,r}\Big[\,a_{\mathbf{p}\,,\,r}\,u_{\mathbf{p}\,,\,r}\,(x) + a^{\dagger}_{\mathbf{p}\,,\,r}\,u^{*}_{\mathbf{p}\,,\,r}\,(x)\,\Big] = \psi^{\dagger}_M\,(x)$$

where the hermitean conjugation refers only to the creation destruction operators acting on the Fock space. Instead we have the expansion of the left Majorana adjoint spinor

$$\bar\psi_M\,(x) = \sum_{\mathbf{p}\,,\,r}\Big[\,a^{\dagger}_{\mathbf{p}\,,\,r}\,\bar u_{\mathbf{p}\,,\,r}\,(x) + a_{\mathbf{p}\,,\,r}\,\bar u^{*}_{\mathbf{p}\,,\,r}\,(x)\,\Big] = \bar\psi^{\dagger}_M\,(x)$$

in which

$$\bar u_{\mathbf{p}\,,\,r}\,(x) \equiv [\,(2\pi)^3\,2\omega_{\mathbf{p}}\,]^{-\frac{1}{2}}\;[\,\tilde u^{\top}_r\,(p)\,]^*\,\gamma^0_M\;\exp\{\,i\omega_{\mathbf{p}}\,t - i\,\mathbf{p}\cdot\mathbf{x}\}$$

Notice that from the normalization

$$(\,u_{\mathbf{p}\,,\,r}\,,\,u_{\mathbf{q}\,,\,s}\,) = \int \mathrm{d}\mathbf{x}\;\bar u_{\mathbf{p}\,,\,r}\,(x)\,\gamma^0_M\,u_{\mathbf{q}\,,\,s}\,(x) = \delta_{rs}\;\delta(\mathbf{p}-\mathbf{q})$$

we can readily obtain the normal modes expansions of the observables involving the Majorana massive spinor field. For example the energy momentum four vector takes the form

$$
\begin{aligned}
P_\mu &= \frac{i}{2}\int \mathrm{d}\mathbf{x}\;\;:\bar\psi_M\,(x)\,\gamma^0_M\,\overleftrightarrow{\partial}_\mu\,\psi_M\,(x): \\
&= \sum_{\mathbf{p}\,,\,r}\,p_\mu\,a^{\dagger}_{\mathbf{p}\,,\,r}\,a_{\mathbf{p}\,,\,r} \qquad\qquad (\,p_0=\omega_{\mathbf{p}}\,)
\end{aligned}
$$

Chapter 5

The Vector Field

5.1 General Covariant Gauges

The quantum theory of a relativistic massive vector wave field has been first developed at the very early stage of the quantum field theory by the rumanian theoretical and mathematical great physicist

Alexandru Proca (Bucarest, 16.10.1897 – Paris, 13.12.1955)
Sur les equations fondamentales des particules elémentaires
Comptes Rendu Acad. Sci. Paris **202** (1936) 1490

Nearly twenty years after, a remarkable generalization of the quantization procedure for a massive vector relativistic wave field was discovered by the swiss theoretician

E.C.G. Stueckelberg,
Théorie de la radiation de photons de masse arbitrairement petite
Helv. Phys. Acta **30** (1957) 209-215

whose quantization method is nowadays known as *the Stueckelberg trick*. The covariant quantization of the radiation gauge field has been pionereed long time ago by Gupta and Bleuler

1. Sen N. Gupta, Proc. Phys. Soc. A**63** (1950) 681

2. K. Bleuler, Helv. Phys. Acta **23** (1950) 567

and further fully developed by Nakanishi and Lautrup

1. Noboru Nakanishi, Prog. Theor. Phys. **35** (1966) 1111; *ibid.* **49** (1973) 640; *ibid.* **52** (1974) 1929; Prog. Theor. Phys. Suppl. No.**51** (1972) 1

2. B. Lautrup, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **35** (1967) No. 11, 1

who completely clarified the subject. As we shall see below, there are many common features shared by the quantum dynamics of the massive and the massless relativistic vector fields, besides some crucial differences. Needless to say, the most important property of the massless vector field theory is its invariance under the so called *gauge transformation of the first kind*

$$A_\mu(x) \quad \mapsto \quad A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$$

where $f(x)$ is an arbitrary real function, its consequence being the exact masslessness[1] of the photon as well as the transversality of its polarizations. Conversely, this local symmetry is not an invariance of the massive vector field theory, so that a third longitudinal polarization indeed appears for the massive vector particles.

In what follow, on the one hand I will attempt to treat contextually the massive and massless cases though, on the other hand, I will likely to focuss the key departures between the two items. The main novelty, with respect to the previously studied scalar and spinor relativistic wave fields, is the appearance of *auxiliary, unphysical ghost field* to setup a general covariant and consistent quantization procedure, as well as the unavoidable presence of a space of the quantum states – the Fock space – with an indefinite metric.

We start from the classical Lagrange density

$$\begin{aligned} \mathcal{L}_{A,B} &= -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} m^2 A^\mu(x) A_\mu(x) \\ &+ A^\mu(x) \partial_\mu B(x) + \frac{1}{2} \xi B^2(x) \end{aligned} \tag{5.1}$$

where $B(x)$ is an auxiliary unphysical scalar field of canonical engeneering dimension $[B] = \mathrm{eV}^2$, while the dimensionless parameter $\xi \in \mathbb{R}$ is named the *gauge fixing parameter*, the abelian field strength being as usual $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, in such a manner that the action results to be Poincaré invariant. The variations with respect to the scalar field B and the vector potential A_μ drive to the Euler-Lagrange equations of motion

$$\begin{cases} \partial_\mu F^{\mu\nu}(x) + m^2 A^\nu(x) + \partial^\nu B(x) = 0 \\ \partial_\mu A^\mu(x) = \xi B(x) \end{cases} \tag{5.2}$$

[1]The present day experimental limit on the photon mass is $m_\gamma < 6 \times 10^{-17}$ eV – see *The Review of Particle Physics* C. Amsler et al., Physics Letters **B667**, 1 (2008) and 2009 partial update for 2010.

Taking the four divergence of the first equation and using the second equation we obtain

$$m^2\,\partial\cdot A\,(x) = -\,\Box B\,(x) = m^2\,\xi\,B\,(x)$$

which shows that the auxiliary field is a free real scalar field that satisfies the Klein-Gordon wave equation with a square mass $\xi\,m^2$, which is positive only for $\xi > 0$. This latter feature makes it apparent the unphysical nature of the auxiliary B-field, for it becomes tachyon-like for negative values of the gauge fixing parameter. Nonetheless, it will become soon clear later on why the introduction of the auxiliary and unphysical B-field turns out to bet very convenient and eventually unavoidable, to the aim of building up a covariant quantization of the real vector field, especially in the massless gauge invariant limit $m^2 \to 0$.

If we write the above equations of motion for the vector potential and the auxiliary scalar field we have

$$\left(\Box + m^2\right) A_\mu\,(x) + (\,1-\xi\,)\;\partial_\mu B\,(x) = 0 \tag{5.3}$$

$$\partial\cdot A\,(x) = \xi\,B\,(x) \tag{5.4}$$

or even more explicitly

$$\left\{ g_{\mu\nu}\left(\Box + m^2\right) \;-\; \left(1-\frac{1}{\xi}\right)\partial_\mu\,\partial_\nu \right\} A^\nu\,(x) = 0 \tag{5.5}$$

$$\left(\Box + m^2\,\xi\right) B\,(x) = 0 \tag{5.6}$$

$$\partial\cdot A\,(x) = \xi\,B\,(x) \tag{5.7}$$

the very last relation being usually named *the subsidiary condition.*

The above system (5.2) of field equations, which includes the subsidiary condition, does nicely simplify for the particular value of the gauge fixing parameter $\xi = 1$: namely,

$$\left.\begin{array}{c} (\Box + m^2)\;A^\mu\,(x) = 0 \\ \partial\cdot A\,(x) = B\,(x) \\ (\Box + m^2)\,B\,(x) = 0 \end{array}\right\} \qquad (\,\xi = 1\,) \tag{5.8}$$

and in the massless limit we come to the d'Alembert wave equation

$$\Box A^\mu\,(x) = 0 = \Box B\,(x) \qquad\qquad \partial\cdot A\,(x) = B\,(x) \tag{5.9}$$

This especially simple and convenient choice of the gauge fixing parameter is named the Feynman gauge. If, instead, we put $\xi = 0$ in the Euler-Lagrange equations (5.2) we get

$$\left(\Box + m^2\right) A^{\mu}(x) + \partial^{\mu} B(x) = 0 \qquad \partial \cdot A(x) = 0 = \Box B(x) \tag{5.10}$$

and in the massless limit

$$\Box A^{\mu}(x) + \partial^{\mu} B(x) = 0 \qquad \partial \cdot A(x) = 0 = \Box B(x) \tag{5.11}$$

This latter choice is known as the Lorentz gauge in classical electrodynamics or the Landau gauge in quantum electrodynamics, or even the *renormalizable gauge* in the massive case. The general case of a finite $\xi \neq 0, 1$ is called the *general covariant gauge*.

The canonical energy momentum tensor is obtained in accordance with Noether theorem

$$\begin{aligned} T_{\mu\nu} &= A_{\mu}\,\partial_{\nu} B - F_{\mu\lambda}\,\partial_{\nu} A^{\lambda} - g_{\mu\nu}\,\mathcal{L}_{A,B} \\ &= A_{\mu}\,\partial_{\nu} B - F_{\mu\lambda}\,F_{\nu}{}^{\lambda} - F_{\mu\lambda}\,\partial^{\lambda} A_{\nu} - g_{\mu\nu}\,\mathcal{L}_{A,B} \\ &= -F_{\mu\lambda}\,F_{\nu\kappa}\,g^{\lambda\kappa} - g_{\mu\nu}\,\mathcal{L}_{A,B} \\ &- \partial^{\lambda}(F_{\mu\lambda}\,A_{\nu}) + A_{\nu}\,\partial^{\lambda} F_{\mu\lambda} + A_{\mu}\,\partial_{\nu} B \end{aligned}$$

and using the equation of motion $\partial^{\lambda} F_{\mu\lambda} = \partial_{\mu} B + m^2 A_{\mu}$ we eventually obtain

$$T_{\mu\nu} \doteq \Theta_{\mu\nu} - \partial^{\lambda}(F_{\mu\lambda}\,A_{\nu})$$

in which the symmetric energy momentum tensor reads

$$\begin{aligned} \Theta_{\mu\nu} &\stackrel{\text{def}}{=} A_{\mu}\,\partial_{\nu} B + A_{\nu}\,\partial_{\mu} B \\ &- g^{\lambda\rho}\,F_{\mu\lambda}\,F_{\nu\rho} + m^2\,A_{\nu}\,A_{\mu} - g_{\mu\nu}\,\mathcal{L}_{A,B} \end{aligned} \tag{5.12}$$

It follows therefrom that the conserved Hamilton functional becomes

$$\begin{aligned} P_0 &= \int \mathrm{d}\mathbf{x}\; T_{00}(t,\mathbf{x}) = \int \mathrm{d}\mathbf{x}\,\Big\{ A_0(t,\mathbf{x})\,\dot{B}(t,\mathbf{x}) + F_{0k}(t,\mathbf{x})\,\dot{A}_k(t,\mathbf{x}) \\ &+ \tfrac{1}{4}\,F^{\rho\sigma}(t,\mathbf{x})\,F_{\rho\sigma}(t,\mathbf{x}) - \tfrac{1}{2}\,m^2\,A^{\lambda}(t,\mathbf{x})\,A_{\lambda}(t,\mathbf{x}) \\ &- A^{\mu}(t,\mathbf{x})\,\partial_{\mu} B(t,\mathbf{x}) - \tfrac{1}{2}\,\xi\,B^2(t,\mathbf{x}) \Big\} \end{aligned}$$

The form of the canonical conjugated momenta can be derived from the Lagrange density (5.1)

$$
\begin{aligned}
\Pi_{\mu}(x) &= \delta\mathcal{L}/\delta\dot{A}^{\mu}(x) = \begin{cases} 0 & \text{for } \mu = 0 \\ F_{k0} \equiv E_k & \text{for } \mu = k = 1,2,3 \end{cases} && (5.13) \\
\Pi(x) &= \delta\mathcal{L}/\delta\dot{B}(x) = A_0(x) && (5.14)
\end{aligned}
$$

Now, if we take into account that

$$
\begin{aligned}
F_{0k}(t,\mathbf{x})\,\dot{A}_k(t,\mathbf{x}) &= F_{0k}(t,\mathbf{x})\,F_{0k}(t,\mathbf{x}) + F_{0k}(t,\mathbf{x})\,\partial_k A_0(t,\mathbf{x}) \\
&= E^k(t,\mathbf{x})\,E^k(t,\mathbf{x}) - A_0(t,\mathbf{x})\,\partial_k E^k(t,\mathbf{x}) \\
&+ \partial_k \Big[F_{0k}(t,\mathbf{x})\,A_0(t,\mathbf{x}) \Big]
\end{aligned}
$$

we can recast the energy, up to an irrelevant spatial divergence, in the form

$$
\begin{aligned}
P_0 &\doteq \int \mathrm{d}\mathbf{x}\,\Big\{ -\tfrac{1}{2}m^2\,\Pi^2(t,\mathbf{x}) - \Pi(t,\mathbf{x})\,\partial_k E^k(t,\mathbf{x}) \\
&+ \tfrac{1}{2}E^k(t,\mathbf{x})\,E^k(t,\mathbf{x}) + \tfrac{1}{4}F_{jk}(t,\mathbf{x})\,F_{jk}(t,\mathbf{x}) - \tfrac{1}{2}\xi\,B^2(t,\mathbf{x}) \\
&+ \tfrac{1}{2}m^2\,A_k(t,\mathbf{x})\,A_k(t,\mathbf{x}) + A_k(t,\mathbf{x})\,\partial_k B(t,\mathbf{x}) \Big\}
\end{aligned}
$$

Now we can rewrite the Euler-Lagrange equations (5.2) in the hamiltonian canonical form, that is

$$
\begin{aligned}
\dot{B}(x) &= \{B(x)\,,\,H\} = -m^2\,\Pi(x) - \partial_k E^k(x) \\
&= -m^2\,A_0(x) - \partial_k F_{0k}(x) && (5.15) \\
\partial_0 A_k(x) &= \{A_k(x)\,,\,H\} = F_{0k}(x) + \partial_k \Pi(x) \\
&= F_{0k}(x) + \partial_k A_0(x) && (5.16) \\
\dot{\Pi}(x) &= \{\Pi(x)\,,\,H\} = \partial_k A_k(x) + \xi\,B(x) && (5.17) \\
\dot{F}_{0k}(x) &= \{E^k(x)\,,\,H\} \\
&= \partial_j F_{jk}(x) - m^2\,A_k(x) - \partial_k B(x) && (5.18)
\end{aligned}
$$

where $H \equiv P_0$ and I used the canonical Poisson brackets among all the independent pairs of canonical variables $(\,\mathbf{A}\,,\,B\,;\,\mathbf{E}\,,\,\Pi\,)$: namely,

$$
\begin{gathered}
\{A^{\jmath}(t,\mathbf{x})\,,\,E^k(t,\mathbf{y})\} = g^{\jmath k}\,\delta(\mathbf{x}-\mathbf{y}) \\
\{B(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \{B(t,\mathbf{x})\,,\,A_0(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \qquad (5.19)
\end{gathered}
$$

all the other Poisson brackets being equal to zero. It is very important to realize that, in the massive case, the hamiltonian functional contains an unusual negative kinetic term $-\frac{1}{2}\,m^2\,\Pi^2\,(t,\mathbf{x})$ for the auxiliary field.

The canonical total angular momentum density follows from the Noether theorem and yields

$$
\begin{aligned}
M^{\,\mu\rho\sigma} \;&\equiv\; x^{\rho}\,T^{\,\mu\sigma} \;-\; x^{\sigma}\,T^{\,\mu\rho} \;+\; S^{\,\mu\rho\sigma} \\
S^{\,\mu\rho\sigma} \;&\stackrel{\mathrm{def}}{=}\; (\delta\mathcal{L}/\delta\partial_{\mu}A_{\nu})\;(-\mathrm{i})\;(S^{\,\rho\sigma})_{\nu\lambda}\,A^{\lambda} \\
&=\; -\,F^{\,\mu\rho}\,A^{\sigma} \;+\; F^{\,\mu\sigma}\,A^{\rho}
\end{aligned}
\tag{5.20}
$$

Hence we find

$$
\begin{aligned}
M^{\,\mu\rho\sigma} \;&=\; x^{\rho}\Big(\Theta^{\mu\sigma} + \partial_{\lambda}\left(F^{\,\lambda\mu}\,A^{\sigma}\right)\Big) \\
&-\; x^{\sigma}\Big(\Theta^{\mu\rho} + \partial_{\lambda}\left(F^{\,\lambda\mu}\,A^{\rho}\right)\Big) \\
&+\; F^{\,\mu\sigma}\,A^{\rho} \;-\; F^{\,\mu\rho}\,A^{\sigma} \\
&=\; x^{\rho}\,\Theta^{\mu\sigma} - x^{\sigma}\,\Theta^{\mu\rho} \\
&+\; \partial_{\lambda}\Big(F^{\,\lambda\mu}\,(x^{\rho}\,A^{\sigma} \;-\; x^{\sigma}\,A^{\rho})\Big)
\end{aligned}
\tag{5.21}
$$

Since the very last term does not contribute to the continuity equation $\partial_{\mu}\,M^{\,\mu\rho\sigma} = 0$, we see that the total angular momentum tensor can be always written in the purely orbital form

$$
M^{\,\rho\sigma} = \int \mathrm{d}\mathbf{x}\;\left[\,x^{\rho}\,\Theta^{0\sigma}\,(x) - x^{\sigma}\,\Theta^{0\rho}\,(x)\,\right]
$$

5.2 Normal Modes Decomposition

To solve the Euler-Lagrange system of equations (5.2) in the general case, it is very convenient to decompose the vector potential according to the definition

$$A^{\mu}(x) \stackrel{\text{def}}{=} V^{\mu}(x) - \begin{cases} m^{-2}\,\partial^{\mu} B(x) & m \neq 0 \\ -\,\xi\,\partial^{\mu}\,\mathcal{D} * B(x) & m = 0 \end{cases} \tag{5.22}$$

where the integro-differential operator $\mathcal{D}$ defined by

$$\mathcal{D} \stackrel{\text{def}}{=} \frac{1}{2}\left(\boldsymbol{\nabla}^{2}\right)^{-1}(x_0\partial_0 - c) \tag{5.23}$$

for arbitrary constant c works as an inverse of the d'Alembert wave operator $\Box$ in front of any solution of the wave equation : namely,

$$\Box\,\mathcal{D} * f(x) = f(x) \qquad \text{if} \quad \Box\, f(x) = 0$$

as it can be readily checked by direct inspection. Hence, the subsidiary condition (5.7) entails the *transversality condition*, *i.e.*

$$\partial_{\mu} A^{\mu} \;=\; \xi\, B \quad \Leftrightarrow \quad \partial_{\mu} V^{\mu} \;=\; 0$$

so that we eventually obtain from the equations of motion (5.4)

$$\begin{cases} (\Box + m^2)\, V_{\mu}(x) = 0 \\ \partial^{\mu} V_{\mu}(x) = 0 \\ (\Box + \xi\, m^2)\, B(x) = 0 \end{cases} \qquad (m \neq 0) \tag{5.24}$$

$$\begin{cases} \Box\, V^{\mu}(x) + \partial^{\mu} B(x) = 0 \\ \partial^{\mu} V_{\mu}(x) = 0 \\ \Box\, B(x) = 0 \end{cases} \qquad (m = 0) \tag{5.25}$$

It is worthwhile to realize that the field strength does not depend upon the unphysical auxiliary field $B(x)$, albeit merely on the transverse vector field $V_{\mu}(x)$ since we have $F_{\mu\nu}(x) = \partial_{\mu} V_{\nu}(x) - \partial_{\nu} V_{\mu}(x)$. This means that we can also write

$$\left(\Box + m^2\right) F_{\mu\nu}(x) = 0 \qquad \text{or} \qquad \Box\, F_{\mu\nu}(x) = 0$$

The transverse real vector field $V_{\mu}(x)$ is also named the *Proca vector field* in the massive case, while it is called the tranverse vector potential in the massless case. We shall now find the general solution of above system of the equations of motion in both cases, *i.e.* , the massive and the massless cases.

5.2.1 Normal Modes of the Massive Vector Field

Let me first discuss the normal mode decomposition in the massive case. To this purpose, if we set

$$V_\mu(x) = (2\pi)^{-3/2} \int \mathrm{d}k\ \widetilde{V}_\mu(k)\ \exp\{-\mathrm{i}\,k\cdot x\} \tag{5.26}$$

$$\widetilde{V}^{\,*}_\mu(k) = \widetilde{V}_\mu(-k)$$

then we find

$$\widetilde{V}_\mu(k) = f_\mu(k)\,\delta\left(k^2 - m^2\right) \qquad\qquad k\cdot f(k) = 0 \tag{5.27}$$

where $f_\mu(k)$ are regular functions on the hyperboloid $k^2 = m^2$, which are any arbitrary functions, but for the transversality condition $k\cdot f(k) = 0$ and the reality condition $f^{\,*}_\mu(k) = f_\mu(-k)$.

Next, it is convenient to introduce the three linear polarization real unit vectors $e^{\,\mu}_r(\mathbf{k})$ $(r = 1,2,3)$ which are defined by the properties

$$k_\mu\, e^{\,\mu}_r(\mathbf{k}) = 0 \qquad\qquad (r = 1,2,3) \qquad k_0 \equiv \omega_{\mathbf{k}} = \left(\mathbf{k}^2 + m^2\right)^{1/2}$$

$$-\,g_{\mu\nu}\, e^{\,\mu}_r(\mathbf{k})\, e^{\,\nu}_s(\mathbf{k}) = \delta_{rs} \qquad\qquad (\text{orthonormality relation}) \tag{5.28}$$

$$\begin{aligned} &\sum_{r=1}^{3} e^{\,\mu}_r(\mathbf{k})\, e^{\,\nu}_r(\mathbf{k}) \\ &= -\,g^{\,\mu\nu} + k^\mu k^\nu / k^2 \\ &= -\,g^{\,\mu\nu} + k^\mu k^\nu / m^2 \qquad\qquad (\text{closure relation}) \end{aligned} \tag{5.29}$$

A suitable explicit choice is provided by

$$\left.\begin{array}{c} \mathrm{e}^{\,0}_r(\mathbf{k}) = 0 \\ \mathbf{k}\cdot\mathbf{e}_r(\mathbf{k}) = 0 \\ \mathbf{e}^*_r(\mathbf{k})\cdot\mathbf{e}_s(\mathbf{k}) = \delta_{rs} \end{array}\right\} \quad \text{for } r,s = 1,2$$

$$\mathrm{e}^{\,0}_3(\mathbf{k}) = \frac{|\,\mathbf{k}\,|}{m} \qquad\qquad \mathbf{e}_3(\mathbf{k}) = \frac{\widehat{\mathbf{k}}}{m}\,\omega_{\mathbf{k}}$$

in such a manner that we can write the normal mode decomposition of the classical Proca real vector field in the form

$$V^{\nu}(x) = \sum_{\mathbf{k},r} \left[f_{\mathbf{k},r}\, u^{\nu}_{\mathbf{k},r}(x) + f^{*}_{\mathbf{k},r}\, u^{\nu\,*}_{\mathbf{k},r}(x) \right] \tag{5.30}$$

$$u^{\nu}_{\mathbf{k},r}(x) = [\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-1/2}\, e^{\nu}_{r}(\mathbf{k})\, \exp\{-\,\mathrm{i}\,\omega_{\mathbf{k}} x^0 + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\} \tag{5.31}$$

where we have denoted as usual

$$\sum_{\mathbf{k},r} \stackrel{\mathrm{def}}{=} \int \mathrm{d}\mathbf{k} \sum_{r=1}^{3}$$

Notice that the set of the vector wave functions $u^{\nu}_{\mathbf{k},r}(x)$ does satisfy the orthonormality and closure relations

$$-\,g_{\lambda\sigma} \int d\mathbf{x}\ u^{\sigma\,*}_{\mathbf{k},s}(y)\,\mathrm{i}\,\overleftrightarrow{\partial}_0\, u^{\lambda}_{\mathbf{h},r}(x) = \delta\,(\mathbf{h}-\mathbf{k})\ \delta_{rs} \tag{5.32}$$

$$\sum_{\mathbf{k},r} u^{\lambda}_{\mathbf{k},r}(x)\, u^{\nu\,*}_{\mathbf{k},r}(y) \;=\; \mathrm{i}\left(g^{\lambda\nu} + m^{-2}\,\partial^{\lambda}_{x}\,\partial^{\nu}_{x}\right)\, D^{(-)}(x-y) \tag{5.33}$$

Next we have

$$B(x) = m \sum_{\mathbf{k}} \left[b_{\mathbf{k}}\, u_{\mathbf{k}}(x) + b^{*}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(x) \right]$$

$$u_{\mathbf{k}}(x) = [\,(2\pi)^3\, 2\,\omega'_{\mathbf{k}}\,]^{-1/2}\ \exp\{-\,\mathrm{i}\,\omega'_{\mathbf{k}} x^0 + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\}$$

$$\omega'_{\mathbf{k}} \;\equiv\; \left(\mathbf{k}^2 + \xi\, m^2\right)^{1/2} \tag{5.34}$$

so that from eq. (5.22) we eventually come to the normal mode decomposition of the classical real vector potential

$$\begin{aligned} A^{\mu}(x) &= \sum_{\mathbf{k},r} \left[f_{\mathbf{k},r}\, u^{\mu}_{\mathbf{k},r}(x) + f^{*}_{\mathbf{k},r}\, u^{\mu\,*}_{\mathbf{k},r}(x) \right] \\ &+ \frac{\mathrm{i}}{m} \sum_{\mathbf{k}} k'^{\mu} \left[b_{\mathbf{k}}\, u_{\mathbf{k}}(x) \;-\; b^{*}_{\mathbf{k}}\, u^{*}_{\mathbf{k}}(x) \right] \\ k'^{\mu} &= (\omega'_{\mathbf{k}}\,,\, \mathbf{k}) \end{aligned} \tag{5.35}$$

As a final remark it is very instructive, in the massive case, to rewrite the classical Lagrange density and the canonical energy momentum tensor as functionals of the transverse Proca vector field $V_{\mu}(x)$ and of the unphysical auxiliary scalar field $B(x)$. Actually, making use of the decomposition (5.22) we obtain

$$
\begin{aligned}
\mathcal{L}_{A,B} &= -\tfrac{1}{4}F^{\mu\nu}(x)\,F_{\mu\nu}(x) + \tfrac{1}{2}m^2\,V^{\mu}(x)\,V_{\mu}(x) \\
&- (1/2m^2)\ \partial^{\mu}B(x)\,\partial_{\mu}B(x) + \tfrac{1}{2}\xi\,B^2(x) \\
&\equiv \mathcal{L}_V + \mathcal{L}_B
\end{aligned} \tag{5.36}
$$

Notice that the lagrangian of the transverse vector field $V^{\mu}(x)$ entails the Euler-Lagrange equations

$$\partial_{\mu}F^{\mu\nu}(x) + m^2\,V^{\nu}(x) = 0$$

so that the transversality condition

$$\partial_{\mu}V^{\mu}(x) = 0$$

does indeed follow from the equations of motion. The canonical conjugate momenta are given by

$$\Pi^{\mu}(x) = \delta\mathcal{L}_V\,/\,\delta\dot{V}_{\mu}(x) = \begin{cases} 0 & \text{for } \mu = 0 \\ F_{0k} = E^k & \text{for } \mu = k = 1,2,3 \end{cases} \tag{5.37}$$

$$\Pi(x) = \delta\mathcal{L}_B\,/\,\delta\dot{B}(x) = -\,m^{-2}\,\dot{B}(x) \tag{5.38}$$

and consequently we get the Poisson brackets

$$\{V_k(t,\mathbf{x})\,,\,E^{\ell}(t,\mathbf{y})\} = \delta_k^{\ell}\,\delta(\mathbf{x}-\mathbf{y}) \tag{5.39}$$

$$\{B(t,\mathbf{x})\,,\,\Pi(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \tag{5.40}$$

all the remaining Poisson brackets being equal to zero. Also the canonical energy momentum second rank tensor can be conveniently expressed in terms of the transverse vector field, up to the irrelevant term $\partial_{\lambda}\left(A_{\nu}\,F^{\mu\lambda}\right)$ *viz.*,

$$
\begin{aligned}
\Theta^{\mu}{}_{\nu} &= \tfrac{1}{4}\delta^{\mu}{}_{\nu}\left(F^{\rho\sigma}F_{\rho\sigma} - 2m^2\,V^{\lambda}V_{\lambda}\right) + m^2\,V_{\nu}\,V^{\mu} - F^{\mu\lambda}F_{\nu\lambda} \\
&- \frac{1}{m^2}\,\partial^{\mu}B\,\partial_{\nu}B + \left(\frac{1}{2m^2}\,\partial^{\lambda}B\,\partial_{\lambda}B - \tfrac{1}{2}\xi\,B^2\right)\delta^{\mu}{}_{\nu}
\end{aligned} \tag{5.41}
$$

which appears to be symmetric and conserved. As a consequence, when the total angular momentum density is expressed as a functional of the transverse vector field $V_{\mu}(x)$ and of the unphysical auxiliary field $B(x)$, *i.e.*

$$M^{\mu\rho\sigma} = x^{\rho}\,\Theta^{\mu\sigma} - x^{\sigma}\,\Theta^{\mu\rho} - F^{\mu\rho}\,V^{\sigma} + F^{\mu\sigma}\,V^{\rho} \tag{5.42}$$

we see that the spin angular momentum density third rank tensor

$$S^{\mu\rho\sigma} = F^{\mu\sigma} V^{\rho} - F^{\mu\rho} V^{\sigma}$$

does satisfy, thanks to the symmetry of $\Theta_{\mu\nu}$, the continuity equation

$$\partial_{\mu} S^{\mu\rho\sigma} = 0$$

so that the spin angular momentum second rank tensor

$$S^{jk} = \int \mathrm{d}\mathbf{x}\ \left(V^{j} E^{k} - E^{j} V^{k} \right) \tag{5.43}$$

$$S^{k0} = \int \mathrm{d}\mathbf{x}\ E^{k}(t,\mathbf{x})\, V_0(t,\mathbf{x}) \tag{5.44}$$

is indeed conserved. As a final comment, I would like to remark that the symmetric form (5.41) of the energy momentum tensor does not admit a well defined massless limit. Thus, a conserved spin angular momentum tensor can no longer be properly defined in the massless case.

5.2.2 Normal Modes of the Gauge Potential

Let me now turn to the discussion of the massless case. Here, it is very important to gather that the transversality condition

$$0 = \partial^{\mu} V_{\mu}(x) = (2\pi)^{-3/2} \int \mathrm{d}k\ (-\mathrm{i})\, k^{\mu}\, \widetilde{V}_{\mu}(k)\ \exp\{-\mathrm{i}\, k\cdot x\} \tag{5.45}$$

$$\widetilde{V}^{*}_{\mu}(k) = \widetilde{V}_{\mu}(-k)$$

on the light-cone $k^2 = 0$ is fulfilled by three independent **linear polarization** unit **real** vectors $\varepsilon^{\mu}_{A}(\mathbf{k})$ $(A = 1, 2, L)$ which are defined by the properties

$$k_{\mu}\, \varepsilon^{\mu}_{A}(\mathbf{k}) = 0 \qquad (A = 1, 2, L) \qquad k_0 \equiv \omega_{\mathbf{k}} = |\,\mathbf{k}\,|$$

$$\left.\begin{array}{c} \varepsilon^{0}_{A}(\mathbf{k}) = 0 \\ \mathbf{k}\cdot\boldsymbol{\varepsilon}_{A}(\mathbf{k}) = 0 \\ \boldsymbol{\varepsilon}_{A}(\mathbf{k})\cdot\boldsymbol{\varepsilon}_{B}(\mathbf{k}) = \delta_{AB} \end{array}\right\} \quad \text{for } A, B = 1, 2$$

$$\varepsilon^{\mu}_{L}(\mathbf{k}) \equiv k^{\mu} = (\,|\,\mathbf{k}\,|\ ,\ \mathbf{k}\,)$$

Now, if we introduce a further light-like polarization vector

$$\begin{aligned} \varepsilon^{\lambda}_{S}(\mathbf{k}) &= \tfrac{1}{2}\,(\,|\,\mathbf{k}\,|\ ,\ -\mathbf{k}\,)\,/\,|\,\mathbf{k}\,|^{2}\ \equiv\ k^{\lambda}_{*}\,/\,k\cdot k_{*} \\ k^{\lambda}_{*} &= (\,|\,\mathbf{k}\,|\ ,\ -\mathbf{k}\,) \qquad k_{\lambda}\cdot\varepsilon^{\lambda}_{S} = 1 \end{aligned} \tag{5.46}$$

then we can write

$$- g_{\mu\nu}\, \varepsilon_A^{\,\mu}(\mathbf{k})\, \varepsilon_B^{\,\nu}(\mathbf{k}) = \eta_{AB} \qquad \text{(orthonormality relation)} \tag{5.47}$$

$$\eta_{AB}\, \varepsilon_A^{\,\mu}(\mathbf{k})\, \varepsilon_B^{\,\nu}(\mathbf{k}) = - g^{\,\mu\nu} \qquad \text{(closure relation)} \tag{5.48}$$

where

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad (A, B = 1, 2, L, S)$$

According to the conventional wisdom, the labels L and S do correspond respectively to the *longitudinal polarization* and the *scalar polarization* of the transverse massless vector particles. Moreover, it is convenient to introduce the transverse projector and the light-cone projector

$$\begin{aligned} \Pi_{\perp}^{\lambda\nu}(k) &= g^{\lambda\nu} - \frac{k^{\lambda} k_*^{\nu} + k^{\nu} k_*^{\lambda}}{k \cdot k_*} \\ &= - \sum_{A=1,2} \varepsilon_A^{\lambda}(\mathbf{k})\, \varepsilon_A^{\nu}(\mathbf{k}) \end{aligned} \tag{5.49}$$

$$\begin{aligned} \Pi_{\vee}^{\lambda\nu}(k) &= \frac{k^{\lambda} k_*^{\nu} + k^{\nu} k_*^{\lambda}}{k \cdot k_*} \\ &= \varepsilon_L^{\lambda}(\mathbf{k})\, \varepsilon_S^{\nu}(\mathbf{k}) + \varepsilon_S^{\lambda}(\mathbf{k})\, \varepsilon_L^{\nu}(\mathbf{k}) \end{aligned} \tag{5.50}$$

which satisfy by construction

$$\Pi_{\perp}^{\lambda\nu}(k) = \left(\Pi_{\perp}^{\nu\lambda}(k)\right)^* \qquad g_{\mu\rho}\, \Pi_{\perp}^{\mu\nu}(k)\, \Pi_{\perp}^{\rho\sigma}(k) = \Pi_{\perp}^{\nu\sigma}(k)$$

$$\Pi_{\vee}^{\lambda\nu}(k) = \left(\Pi_{\vee}^{\nu\lambda}(k)\right)^* \qquad g_{\mu\rho}\, \Pi_{\vee}^{\mu\nu}(k)\, \Pi_{\vee}^{\rho\sigma}(k) = \Pi_{\vee}^{\nu\sigma}(k)$$

$$\mathrm{tr}\, \Pi_{\perp}(k) = g_{\lambda\nu}\, \Pi_{\perp}^{\lambda\nu}(k) = 2 = \mathrm{tr}\, \Pi_{\vee}(k) = g_{\lambda\nu}\, \Pi_{\vee}^{\lambda\nu}(k)$$

$$\Pi_{\perp}^{\mu\nu}(k) + \Pi_{\vee}^{\mu\nu}(k) = g^{\mu\nu}$$

as it can be readily verified by direct inspection. As a consequence, for any given light-like momentum k^{μ} $(k^2 = 0)$ the *physical photon polarization density matrix*

$$\rho_{\perp}^{\lambda\nu}(k) \stackrel{\text{def}}{=} -\tfrac{1}{2} \sum_{A=1,2} \varepsilon_A^{\lambda}(\mathbf{k})\, \varepsilon_A^{\nu}(\mathbf{k}) \tag{5.51}$$

$$\rho_{\perp}^{\lambda\nu}(k) = \left(\rho_{\perp}^{\nu\lambda}(k)\right)^* \qquad \mathrm{tr}\, \rho_{\perp} = 1 \tag{5.52}$$

will represent a mixed state corresponding to unpolarized monochromatic photons. In conclusion, we can finally write the normal mode decomposition of the classical real transverse massless vector field in the form

$$
\begin{aligned}
V^{\lambda}(x) &= \sum_{\mathbf{k},A} \left[\, g_{\mathbf{k},A}\, u^{\lambda}_{\mathbf{k},A}(x) + g^{*}_{\mathbf{k},A}\, u^{\lambda *}_{\mathbf{k},A}(x) \,\right] \\
&- \partial^{\,\lambda}\, \mathcal{D} * B(x)
\end{aligned}
\tag{5.53}
$$

$$
u^{\lambda}_{\mathbf{k},A}(x) = [\,(2\pi)^3\, 2\,|\,\mathbf{k}\,|\,]^{-1/2}\, \varepsilon^{\,\lambda}_{A}(\mathbf{k})\, \exp\{-\,\mathrm{i}\, k\cdot x\}
$$

$$
A = 1, 2, L, S \qquad\qquad k_0 = |\,\mathbf{k}\,|
$$

together with

$$
\begin{aligned}
B(x) &= \partial_{\lambda} \sum_{\mathbf{k},A} \left[\, g_{\mathbf{k},A}\, u^{\lambda}_{\mathbf{k},A}(x) + g^{*}_{\mathbf{k},A}\, u^{\lambda *}_{\mathbf{k},A}(x) \,\right] \\
&= \frac{1}{\mathrm{i}} \sum_{\mathbf{k}} \left[\, k\cdot u_{\mathbf{k},S}(x)\, g_{\mathbf{k},S} - \, k\cdot u^{*}_{\mathbf{k},S}(x)\, g^{*}_{\mathbf{k},S} \,\right]
\end{aligned}
\tag{5.54}
$$

The real transverse massless vector field $V^{\lambda}(x)$ is also named the vector potential in the Landau gauge $\xi = 0$. Then, from eq. (5.22) we eventually come to the normal mode decomposition of the classical real massless vector potential

$$
\begin{aligned}
A^{\lambda}(x) &= V^{\lambda}(x) + \xi\, \partial^{\,\lambda}\, \mathcal{D} * B(x) \\
&= \sum_{\mathbf{k},A} \left[\, g_{\mathbf{k},A}\, u^{\lambda}_{\mathbf{k},A}(x) + g^{*}_{\mathbf{k},A}\, u^{\lambda *}_{\mathbf{k},A}(x) \,\right] \\
&- (1-\xi)\, \partial^{\,\lambda}\, \mathcal{D} * B(x)
\end{aligned}
\tag{5.55}
$$

Notice that the complete and orthonormal system of the positive frequency plane wave solutions $u^{\lambda}_{\mathbf{k},A}(x)$ for the massless gauge vector potential does satisfy

$$
-\, g_{\lambda\nu} \int \mathrm{d}\mathbf{x}\; u^{\lambda *}_{\mathbf{h},A}(x)\, i\, \overleftrightarrow{\partial}_0\, u^{\nu}_{\mathbf{k},B}(x) = \delta\,(\mathbf{h}-\mathbf{k})\; \eta_{AB} \tag{5.56}
$$

$$
\eta_{AB} \sum_{\mathbf{k}} \; u^{\lambda}_{\mathbf{k},A}(x)\, u^{\nu *}_{\mathbf{k},B}(y) = \frac{1}{\mathrm{i}}\, g^{\,\lambda\nu}\, \triangle^{(-)}(x-y) \tag{5.57}
$$

$$
\sum_{\mathbf{k}} \; u^{\lambda}_{\mathbf{k},L}(x)\, u^{\nu *}_{\mathbf{k},S}(y) = \frac{1}{\mathrm{i}}\, \partial^{\,\lambda}_{x}\, \partial^{\,\nu}_{*y}\; \triangle^{(-)}(x-y) \tag{5.58}
$$

where the massless scalar positive frequency distribution is given by

$$\triangle^{(-)}(x) = \frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}k\ \delta\left(k^2\right)\ \theta\left(k_0\right)\ \exp\{-\,i\,k\cdot x\}$$

whereas I have set

$$\partial_x^\lambda\,\partial_{*\,y}^\nu\,\triangle^{(-)}(x-y) \equiv \frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}k\ \frac{k^\lambda k_*^\nu}{k\cdot k_*}\ \delta\left(k^2\right)\ \theta\left(k_0\right)\ \exp\{-\,i\,k\cdot x\}$$

We are now ready to perform the general covariant canonical quantization of the free vector fields, both in the massive and massless cases.

5.3 Covariant Canonical Quantization

The manifestly covariant canonical quantization of the massive and massless real vector free fields can be done in a close analogy with the canonical quantization of the real scalar free field, just like I have done in Section 3.2.

5.3.1 The Massive Vector Field

Let us first consider the tranverse vector free fields. In the massive case, the classical Proca vector free field (5.31) is turned into an operator valued tempered distribution : namely

$$V^{\nu}(x) = \sum_{\mathbf{k}\,,\,r} \Big[\, f_{\mathbf{k}\,,\,r}\, u^{\nu}_{\mathbf{k}\,,\,r}(x) + f^{\dagger}_{\mathbf{k}\,,\,r}\, u^{\nu\,*}_{\mathbf{k}\,,\,r}(x) \Big] \tag{5.59}$$

$$u^{\nu}_{\mathbf{k}\,,\,r}(x) = [\,(2\pi)^3\, 2\omega_{\mathbf{k}}\,]^{-1/2}\, e^{\mu}_{r}(\mathbf{k})\, \exp\{-\,\mathrm{i}\,\omega_{\mathbf{k}}\, x^0 + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}\} \tag{5.60}$$

where the creation annihilation operators satisfy the canonical commutation relations

$$[\, f_{\mathbf{h}\,,\,r}\,,\, f_{\mathbf{k}\,,\,s}\,] = 0 = [\, f^{\dagger}_{\mathbf{h}\,,\,r}\,,\, f^{\dagger}_{\mathbf{k}\,,\,s}\,]$$

$$[\, f_{\mathbf{h}\,,\,r}\,,\, f^{\dagger}_{\mathbf{k}\,,\,s}\,] = \delta_{rs}\, \delta(\mathbf{h}-\mathbf{k})$$

As a consequence, the normal mode expansion of the field strength becomes

$$\begin{aligned} \mathbf{E}(x) &= \mathrm{i} \sum_{\mathbf{k}\,,\,r} \Big\{ f_{\mathbf{k}\,,\,r}\, \big[\,\omega_{\mathbf{k}}\, \mathbf{u}_{\mathbf{k}\,,\,r}(x) - \mathbf{k}\, u^{0}_{\mathbf{k}\,,\,r}(x)\,\big] \\ &- f^{\dagger}_{\mathbf{k}\,,\,r}\, \big[\,\omega_{\mathbf{k}}\, \mathbf{u}^{*}_{\mathbf{k}\,,\,r}(x) - \mathbf{k}\, u^{0\,*}_{\mathbf{k}\,,\,r}(x)\,\big] \Big\} \qquad (5.61) \\ \mathbf{B}(x) &= \mathrm{i} \sum_{\mathbf{k}\,,\,r} \mathbf{k} \times \mathbf{u}_{\mathbf{k}\,,\,r}(x)\, f_{\mathbf{k}\,,\,r} \;+\; \mathrm{c.c.} \qquad (5.62) \end{aligned}$$

where the massive electric and massive magnetic fields are defined by

$$E^{\,k} = F^{\,k0} = -\,\big(\partial_0 A^{\,k} + \partial_k A_0\big) \qquad\qquad B^{\,k} = \tfrac{1}{2}\,\varepsilon^{\,j\ell k}\, F_{j\ell}$$

It is in fact a straightforward exercise to show that, by inserting the normal mode expansion (5.60) and by making use of the orthonormality relations among the polarization vectors $e^{\mu}_{r}(\mathbf{k})$, the energy momentum operator takes the expected diagonal form, which corresponds to the sum over an infinite set of independent linear harmonic oscillators, one for each component of the wave vector $\mathbf{k}$ and for each one of the three independent physical polarization $e_r^{\mu}(\mathbf{k})$ $(r = 1,2,3)$. Actually, if we recast the equations of motion (5.24) for the Proca wave field in the Maxwell-like form

$$\begin{cases} \partial_j F_{jk} = \dot{F}_{0k} + m^2 V_k & \text{(displacement current)} \\ \partial_k F_{0k} = -\, m^2 V_0 & \text{(Gauss law)} \\ \dot{V}_0 = \partial_k V_k & \text{(subsidiary condition)} \end{cases} \tag{5.63}$$

then we get the quite simple expression for the energy operator

$$\begin{aligned} P_0 &= H = \int \mathrm{d}\mathbf{x}\ :\Theta_{00}(t,\mathbf{x}): \\ &= \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ : E^k(x)\,E^k(x) + \tfrac{1}{2}\,F_{jk}(x)\,F_{jk}(x) : \\ &+ \tfrac{1}{2}\,m^2 \int \mathrm{d}\mathbf{x}\ : V_0^2(x) + V_k(x)\,V_k(x) : \\ &\doteq \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ : F_{0k}(x)\,\overleftrightarrow{\partial}_0\,V_k(x) : \\ &\doteq \int \mathrm{d}\mathbf{x}\ : \tfrac{1}{2}\,V_\mu(x)\,\overleftrightarrow{\partial}_0\,\dot{V}^{\mu}(x) : \end{aligned}$$

where $\doteq$ means, as usual, that I have dropped some spatial divergence term and I have repeatedly made use of the equations of motion. Now, from the orthonormality relation (5.28) we can easily recognize that

$$g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\ u_{\mathbf{k},r}^{\lambda\,*}(x)\,\overleftrightarrow{\partial}_0\,\dot{u}_{\mathbf{h},s}^{\nu}(x) = \omega_{\mathbf{k}}\,\delta_{rs}\,\delta(\mathbf{h}-\mathbf{k})$$

$$g_{\lambda\nu}\int \mathrm{d}\mathbf{x}\ u_{\mathbf{k},r}^{\lambda}(x)\,\overleftrightarrow{\partial}_0\,\dot{u}_{\mathbf{h},s}^{\nu}(x) \equiv 0$$

and consequently

$$\begin{aligned} P_0 &= \sum_{\mathbf{k},r} \omega_{\mathbf{k}}\, f_{\mathbf{k},r}^{\dagger}\, f_{\mathbf{k},r} \\ \mathbf{P} &= \int \mathrm{d}\mathbf{x}\ : \mathbf{E}(x) \times \mathbf{B}(x) + m^2\, V_0(x)\,\mathbf{V}(x) : \\ &\doteq \int \mathrm{d}\mathbf{x}\ : \tfrac{1}{2}\,V_\mu(x)\,\overleftrightarrow{\partial}_k\,\dot{V}^{\mu}(x) : \\ &= \sum_{\mathbf{k},r} \mathbf{k}\, f_{\mathbf{k},r}^{\dagger}\, f_{\mathbf{k},r} \end{aligned} \tag{5.64}$$

On the other hand, the energy momentum tensor of the auxiliary field is provided by the B-dependent part of the general expression (5.41) : namely,

$$\Theta_B^{\mu\nu} \;=\; m^{-2} \;:\; \tfrac{1}{2}\, g^{\mu\nu}\, \partial^{\lambda} B\, \partial_{\lambda} B - \tfrac{1}{2}\, \xi m^2\, g^{\mu\nu}\, B^2 - \partial^{\mu} B\, \partial^{\nu} B \;:$$

which drives to the conserved energy momentum operators

$$P_0 = \frac{1}{2m^2} \int \mathrm{d}\mathbf{x} \;:\; B(x)\, \ddot{B}(x) - \dot{B}^2(x) \;: \tag{5.65}$$

$$\mathbf{P} = \frac{1}{m^2} \int \mathrm{d}\mathbf{x} \;:\; \dot{B}(x)\, \boldsymbol{\nabla} B(x) \;: \tag{5.66}$$

It is important to gather that from the Lagrange density (5.36) it follows that the conjugate momentum of the auxiliary field gets the wrong sign, that means

$$\Pi(x) \;=\; -\, \dot{B}(x)/m^2$$

From the normal mode decomposition of the auxiliary unphysical field and its conjugate momentum

$$B(x) = m \sum_{\mathbf{h}} \left[\, b_{\mathbf{h}}\, u_{\mathbf{h}}(x) + b_{\mathbf{h}}^{\dagger}\, u_{\mathbf{h}}^{*}(x) \right]$$

$$\Pi(y) = \frac{1}{m} \sum_{\mathbf{k}} \; \mathrm{i}\, \omega_{\mathbf{k}}^{\prime} \left[\, b_{\mathbf{k}}\, u_{\mathbf{k}}(y) - b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(y) \right]$$

$$u_{\mathbf{k}}(x) = [\,(2\pi)^3\, 2\, \omega_{\mathbf{k}}^{\prime}\,]^{-1/2}\, \exp\{-\,\mathrm{i}\, \omega_{\mathbf{k}}^{\prime} x_0 + \mathrm{i}\, \mathbf{k}\cdot\mathbf{x}\}$$

$$\omega_{\mathbf{k}}^{\prime} \;\equiv\; \left(\mathbf{k}^2 + \xi\, m^2\right)^{1/2}$$

it is evident that in order to recover the canonical commutation relation

$$[\, B(t,\mathbf{x}) \,,\, \Pi(t,\mathbf{y})\,] = \mathrm{i}\, \delta\,(\mathbf{x} - \mathbf{y})$$

that corresponds to the classical Poisson bracket (5.40) we have to require

$$[\, b_{\mathbf{h}}^{\dagger} \,,\, b_{\mathbf{k}}\,] \;=\; \delta\,(\mathbf{h} - \mathbf{k}) \tag{5.67}$$

all the other commutators vanishing.

`Proof.` We find

$$\begin{aligned}
[\, B(t,\mathbf{x}) \,,\, \Pi(t,\mathbf{y})\,] &= \sum_{\mathbf{h}} \sum_{\mathbf{k}} \mathrm{i}\, \omega_{\mathbf{k}}^{\prime} \left[\, b_{\mathbf{h}}\, u_{\mathbf{h}}(t,\mathbf{x}) + b_{\mathbf{h}}^{\dagger}\, u_{\mathbf{h}}^{*}(t,\mathbf{x}) \,,\, b_{\mathbf{k}}\, u_{\mathbf{k}}(t,\mathbf{y}) - b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(t,\mathbf{y}) \right] \\
&= \sum_{\mathbf{h}} \sum_{\mathbf{k}} \mathrm{i}\, \omega_{\mathbf{k}}^{\prime} \left\{ [\, b_{\mathbf{h}} \,,\, b_{\mathbf{k}}\,]\, u_{\mathbf{h}}(t,\mathbf{x})\, u_{\mathbf{k}}(t,\mathbf{y}) - [\, b_{\mathbf{h}} \,,\, b_{\mathbf{k}}^{\dagger}\,]\, u_{\mathbf{h}}(t,\mathbf{x})\, u_{\mathbf{k}}^{*}(t,\mathbf{y}) \right\} \\
&+ \sum_{\mathbf{h}} \sum_{\mathbf{k}} \mathrm{i}\, \omega_{\mathbf{k}}^{\prime} \left\{ [\, b_{\mathbf{h}}^{\dagger} \,,\, b_{\mathbf{k}}\,]\, u_{\mathbf{h}}^{*}(t,\mathbf{x})\, u_{\mathbf{k}}(t,\mathbf{y}) - [\, b_{\mathbf{h}}^{\dagger} \,,\, b_{\mathbf{k}}^{\dagger}\,]\, u_{\mathbf{h}}^{*}(t,\mathbf{x})\, u_{\mathbf{k}}^{*}(t,\mathbf{y}) \right\}
\end{aligned}$$

so that, if we assume

$$[\,b_{\mathbf{h}}\,,\,b_{\mathbf{k}}^{\dagger}\,] = \,-\,\delta\,(\mathbf{h}-\mathbf{k}) \qquad\quad [\,b_{\mathbf{h}}\,,\,b_{\mathbf{k}}\,] = 0 \qquad (\,\forall\,\mathbf{h},\mathbf{k}\in\mathbb{R}^3\,)$$

then we obtain

$$\begin{aligned}[\,B\,(t,\mathbf{x})\,,\,\Pi\,(t,\mathbf{y})\,] &= \sum_{\mathbf{k}}\mathrm{i}\,\omega'_{\mathbf{k}}\left\{u^*_{\mathbf{k}}\,(t,\mathbf{x})\,u_{\mathbf{k}}\,(t,\mathbf{y}) + u_{\mathbf{k}}\,(t,\mathbf{x})\,u^*_{\mathbf{k}}\,(t,\mathbf{y})\right\} \\ &= \mathrm{i}\int\frac{\mathrm{d}\mathbf{k}}{2(2\pi)^3}\Big(\exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\} + \exp\{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})\}\Big) = \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y})\end{aligned}$$

Thus we can eventually write

$$\begin{aligned} P_0 &= -\sum_{\mathbf{k}}\,\left(\mathbf{k}^2+\xi\,m^2\right)^{1/2}\,b^{\dagger}_{\mathbf{k}}\,b_{\mathbf{k}} \;\equiv\; H_B \qquad (5.68) \\ \mathbf{P} &= \sum_{\mathbf{k}}\,\mathbf{k}\,b^{\dagger}_{\mathbf{k}}\,b_{\mathbf{k}} \end{aligned}$$

From the above expression (5.68) for the conserved hamiltonian of the auxiliary $B-$field, as well as from the unconventional nature of the canonical commutation relations (5.67), it turns out that no physical meaning can be assigned to the hamiltonian operator of the auxiliary scalar field. As a matter of fact, for $\xi\ge 0$ the hamiltonian operator becomes negative definite and unbounded from below, while for $\xi<0$ we se that at low momenta $\mathbf{k}^2 < \mid\xi\mid m^2$ the energy becomes imaginary. In all cases, any physical interpretation of the hamiltonian operator breaks down.

Moreover, from the conventional definition of the Fock vacuum

$$f_{\mathbf{k},r}\,|\,0\,\rangle = 0 \qquad\quad b_{\mathbf{k}}\,|\,0\,\rangle = 0 \qquad\quad \forall\,\mathbf{k}\in\mathbb{R}^3 \quad r=1,2,3$$

and the unconventional canonical commutation relations (5.67), it follows that *e.g.* the proper 1-particle states of the auxiliary field

$$|\,b\rangle \stackrel{\mathrm{def}}{=} \int\mathrm{d}\mathbf{k}\;\tilde{b}\,(\mathbf{k})\,b^{\dagger}_{\mathbf{k}}\,|\,0\,\rangle \qquad\quad \int\mathrm{d}\mathbf{k}\;|\,\tilde{b}\,(\mathbf{k})\,|^2 = 1$$

do exhibit *negative norm*, that is

$$\langle\,b\,|\,b\,\rangle \;=\; -\,1$$

Hence the Fock space $\mathcal{F}$ of the quantum states for the free massive vector field in the general covariant gauge is equipped with an *indefinite metric*, which means that it contains normalizable states with positive, negative and null norm. The auxiliary $B-$field is named a *ghost field* : its presence entails a hamiltonian operator which is unbounded from below, leading thereby to the instability, or even imaginary energy eigenvalues, that means metastable states when $\exp\{-\mathrm{i}\,\omega'_{\mathbf{k}}x_0\}$ is less than one, or even runaway solutions when $\exp\{-\mathrm{i}\,\omega'_{\mathbf{k}}x_0\}$ becomes very large.

Thus, in order to ensure some meaningful and sound quantum mechanical interpretation of the free field theory of a massive vector field in the general covariant gauge, we are necessarily led to select a physical subspace $\mathcal{H}_{\rm phys}$ of the large Fock space $\mathcal{F}$, in which no quanta of the auxiliary field are allowed. This can be achieved by imposing the subsidiary condition

$$B^{(-)}(x)\,|\,\mathrm{phys}\,\rangle \;=\; 0 \qquad\qquad \forall\,|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\rm phys} \tag{5.69}$$

where $B^{(-)}(x)$ denotes, as usual, the positive frequency destruction part of the auxiliary scalar Klein-Gordon ghost field *i.e.*

$$B^{(-)}(x) = m\sum_{\mathbf{k}}\; b_{\mathbf{k}}\,u_{\mathbf{k}}(x)$$

This entails that the vacuum state is physical and cyclic, in such a manner that all the physical states are generated by the repeted action of the massive Proca field creation operators $f^{\dagger}_{\mathbf{k},r}$ on the vacuum.

From the normal mode decompositions of the massive Proca real vector field and of the ghost field, taking the canonical commutation relations into account, we can readily check that we have

$$\begin{aligned}
[\,V_\mu(x)\,,\,V_\nu(y)\,] \;&=\; \mathrm{i}\,\left(g_{\mu\nu}+m^{-2}\,\partial_\mu\,\partial_\nu\right)D\,(x-y\,;m) && (5.70)\\
[\,V_\mu(x)\,,\,B\,(y)\,] \;&=\; 0 && (5.71)\\
[\,B\,(x)\,,\,B\,(y)\,] \;&=\; \mathrm{i}\,m^2\,D\,(x-y\,;\xi\,m) && (5.72)
\end{aligned}$$

the first commutator being due to the closure relation (5.29). In a quite similar way it is very easy to check that the following Feynman propagators actually occur : namely,

$$\begin{aligned}
\langle\,0\,|\,T\,(V_\mu(x)\,V_\nu(y))\,|\,0\,\rangle &= -\left(g_{\mu\nu}+m^{-2}\,\partial_\mu\,\partial_\nu\right)D_F\,(x-y\,;m)\\
\langle\,0\,|\,T\,(B\,(x)\,B\,(y))\,|\,0\,\rangle &= -\,m^2\,D_F\,(x-y\,;\xi\,m)
\end{aligned}$$

Now it is a very simple exercise to obtain the canonical commutator and the Feynman propagator for the vector potential A_μ, taking the basic definition (5.22) into account. Actually, for the Feynman propagator, *e.g.* , we find the Fourier representation

$$\begin{aligned} D^F_{\mu\nu}\,(x\,;\,m\,,\,\xi\,) \;&=\; \langle\,0\,|\,T\,A_\mu\,(x)\,A_\nu\,(0)\,|\,0\,\rangle \qquad (5.73)\\ &=\; \frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}k\;\exp\{-\,\mathrm{i}\,k\cdot x\}\\ &\times\;\left\{\frac{-\,g_{\mu\nu}+m^{-2}\,k_\mu\,k_\nu}{k^2-m^2+\mathrm{i}\,\varepsilon}\;-\;\frac{m^{-2}\,k_\mu\,k_\nu}{k^2-\xi\,m^2+\mathrm{i}\,\varepsilon'}\right\}\\ &=\;\frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}k\;\frac{\exp\{-\,i\,k\cdot x\}}{k^2-m^2+i\,\varepsilon}\left\{-\,g_{\mu\nu}+\frac{(\,1-\xi\,)\,k_\mu\,k_\nu}{k^2-\xi\,m^2+\mathrm{i}\,\varepsilon'}\right\} \end{aligned}$$

which is the celebrated *Stueckelberg propagator*, together with

$$\begin{aligned} \langle\,0\,|\,T\,A_\mu\,(x)\,B\,(y)|\,0\,\rangle &= \partial_\mu\,D_F\,(x-y\,;\,\xi\,m)\\ \langle\,0\,|\,T\,B\,(x)\,B\,(y)\;|\,0\,\rangle &= -\,m^2\,D_F\,(x-y\,;\,\xi\,m) \end{aligned}$$

An important comment is now in order. In the general covariant gauge, for any finite value $\xi\in\mathbb{R}$ of the gauge fixing parameter, the leading asymptotic behaviour for large momenta of the momentum space Feynman propagator is provided by

$$\begin{aligned} \widetilde{D}^F_{\mu\nu}\,(k\,;\,\xi) \;&=\; \frac{1}{k^2-m^2+\mathrm{i}\,\varepsilon}\;\left\{-\,g_{\mu\nu}+\frac{(\,1-\xi\,)\,k_\mu\,k_\nu}{k^2-\xi\,m^2+\mathrm{i}\,\varepsilon'}\right\}\\ &\sim\; k^{-2}\,d_{\mu\nu} \qquad\qquad (\,|\,k_\mu\,|\;\to\;\infty\,) \qquad (5.74) \end{aligned}$$

where $d_{\mu\nu}$ is a constant 4×4 matrix, that is a momentum space isotropic and scale homogeneous quadratically decreasing law. On the one hand, this *naïve* power counting property will become one of the crucial necessary though not sufficient hypotesis in all the up to date available proofs of the perturbative order by order renormalizability of any interacting quantum field theory. On the other hand, the price to be paid is the unavoidable introduction of an auxiliary unphysical $B-$field, that must be eventually excluded from the physical sector of the theory, in such a manner to guarantee the standard orthodox quantum mechanical interpretation. In the free field theory, the subsidiary condition (5.69) is what we need to remove the unphysical quanta. However, it turns out that an extension of the subsidiary condition to the interacting theories appears to be, in general, highly nontrivial.

Hence, the crucial issue of the decoupling of the auxiliary field from the physical sector of an interacting theory involving massive vector field will result to be one of the the most selective and severe model building criterion for a perturbatively renormalizable interacting quantum field theory. In other words, this means that if we consider the scattering operator S of the theory, then, for any pair of physical states $|\,\mathrm{phys}\,\rangle$ and $|\,\mathrm{phys}'\,\rangle$, the *unitarity relation*

$$\langle\,\mathrm{phys}\,|\,S^{\dagger}S\,|\,\mathrm{phys}'\,\rangle = \sum_{\imath}\,\langle\,\mathrm{phys}\,|\,S^{\dagger}\,|\,\imath\,\rangle\,\langle\,\imath\,|\,S\,|\,\mathrm{phys}'\,\rangle = \langle\,\mathrm{phys}\,|\,\mathrm{phys}'\,\rangle$$

must be saturated by a complete orthonormal set of physical intermediate states or, equivalently, the contribution of the unphysical states must cancel in the sum over intermediate states. This unitarity criterion will guarantee the existence of a *well-defined unitary restriction* of the scattering operator to the physical subspace $\mathcal{H}_{\mathrm{phys}} \subset \mathcal{F}$ of the whole Fock space, thus allowing a consistent physical interpretation of the theory.

To this concern, it is important to remark that in the limit $\xi \to \infty$, the auxiliary $B-$field just disappears, so that we are left with the free Proca field, the quanta of which do carry just the three physical polarizations. However, the ultraviolet leading behaviour of the corresponding Feynman propagator becomes

$$\lim_{\xi\to\infty}\ \widetilde{D}^{\,F}_{\mu\nu}(k\,;\xi) = \frac{1}{k^2 - m^2 + \mathrm{i}\,\varepsilon}\ \left\{-\,g_{\mu\nu} + m^{-2}\,k_{\mu}\,k_{\nu}\right\} \tag{5.75}$$

The lack of scale homogeneity and *naïve* power counting property of this expression is the evident obstacle that makes the proof of perturbative order by order renormalizability beyond the present day capabilities. In turn, this is the ultimate reason why the spontaneous gauge symmetry breaking and the Higgs mechanism, to provide the masses for the vector fields which mediate the weak interaction[2], still nowadays appear to be the best buy solution of the above mentioned renormalizabilty versus unitarity issue.

5.3.2 The Gauge Vector Potential

The quantization of the massless gauge vector potential (5.55) is obtained from the canonical commutation relations

$$\left[\,g_{\mathbf{h}\,,A}\,,\,g^{\dagger}_{\mathbf{k}\,,B}\,\right] = \delta\,(\mathbf{h}-\mathbf{k})\ \eta_{AB} \tag{5.76}$$

[2]The weak interaction is mediated by two charged complex vector fields $W^{\pm}$ with mass $M_{\pm} = 80.425 \pm 0.038$ GeV and a neutral real vector field Z^0 with a mass $M_0 = 91.1876 \pm 0.0021$ GeV.

all other commutators vanishing. As a matter of fact, from the normal mode expansion (5.55) and the orthonormality and closure relations (5.56–5.58), we readily obtain the canonical commutation relations

$$\left[\, A^{\lambda}(x)\,,\, A^{\nu}(y)\,\right] = \mathrm{i}\, g^{\lambda\nu}\,\triangle\,(x-y) + \mathrm{i}\;(1-\xi\,)\;\partial_x^{\lambda}\,\partial_y^{\nu}\,E\,(x-y) \tag{5.77}$$

$$[\,B\,(x)\,,\,A^{\nu}\,(y)\,] \;=\; \mathrm{i}\partial_x^{\nu}\,\triangle\,(x-y) \tag{5.78}$$

$$[\,B\,(x)\,,\,B\,(y)\,] \;=\; 0 \tag{5.79}$$

where the massless Pauli-Jordan real and odd distribution is as usual

$$\begin{gathered}
\triangle\,(x) = \triangle^{(-)}\,(x) + \triangle^{(+)}\,(x) = \lim_{m\to 0}\; D\,(x\,;\,m) \\
\triangle^{(\pm)}\,(x) \equiv\; \pm\,\frac{1}{\mathrm{i}}\int \frac{\mathrm{d}\,k}{(2\pi)^3}\;\delta\left(k^2\right)\;\theta\,(k_0\,)\;\exp\{\pm\,\mathrm{i}\,k\cdot x\} \\
\lim_{x_0\to 0}\;\triangle\,(x) = 0 \qquad\qquad \lim_{x_0\to 0}\;\partial_0\,\triangle\,(x) = \delta\,(\mathbf{x}) \\
\triangle\,(x) = \triangle^{*}\,(x) = \;-\,\triangle\,(-x)
\end{gathered}$$

whereas $E\,(x)$ is named the *massless dipole ghost invariant distribution* and is defined by the property

$$\Box\,E\,(x) = \triangle\,(x) \tag{5.80}$$

An explicit representation is provided by

$$\begin{aligned}
E\,(x) \;&=\; \frac{1}{2}\,\left(\boldsymbol{\nabla}^2\right)^{-1}(x_0\,\partial_0 - 1)\;\triangle\,(x) \\
&=\; -\,\lim_{m\to 0}\;\frac{\partial}{\partial\,m^2}\,D\,(x\,;\,m)
\end{aligned} \tag{5.81}$$

and it is an easy task to prove the following useful formula

$$\partial_x^{\,\mu}\,\partial_x^{\,\nu}\,E\,(x-y) = \left(\partial_x^{\,\mu}\,\partial_{*\,x}^{\,\nu} + \partial_y^{\,\nu}\,\partial_{*\,y}^{\,\mu}\right)\triangle\,(x-y)$$

It is a nice exercise to verify the compatibility between the canonical commutation reations (5.79) and the equations of motion

$$\left\{g_{\,\mu\nu}\,\Box\;-\;\left(1-\frac{1}{\xi}\right)\,\partial_{\mu}\,\partial_{\nu}\right\}\,A^{\nu}\,(x) = 0 \tag{5.82}$$

$$\Box\,B\,(x) = 0 \tag{5.83}$$

$$\partial\cdot A\,(x) = \xi\,B\,(x) \tag{5.84}$$

Moreover we find

$$\left[F^{\lambda\rho}(x)\, ,\, A^{\nu}(y) \right] = \left(g^{\nu\rho}\, \mathrm{i}\partial_x^{\lambda} - g^{\lambda\nu}\, \mathrm{i}\partial_x^{\rho} \right) \triangle(x-y) \tag{5.85}$$

that implies

$$\begin{aligned} \left[A^{k}(t,\mathbf{x})\, ,\, E^{\ell}(t,\mathbf{y}) \right] &= \mathrm{i}\, g^{k\ell}\, \delta(\mathbf{x}-\mathbf{y}) \\ \left[B(t,\mathbf{x})\, ,\, \Pi(t,\mathbf{y}) \right] = \mathrm{i}\, \delta(\mathbf{x}-\mathbf{y}) &= \left[B(t,\mathbf{x})\, ,\, A_0(t,\mathbf{y}) \right] \end{aligned}$$

all the remaining equal-time commutation relations being equal to zero, in full agreement with the Bohr, Sommerfeld and Dirac correspondence principle and the classical Poisson brackets (5.19).

A further very important canonical commutation relation is

$$\left[F^{\rho\lambda}(x)\, ,\, B(y) \right] = 0 \tag{5.86}$$

which tells us that a physical local operator such as the electromagnetic field strength tensor indeed commutes with the unphysical auxiliary field for any spacetime separation. Actually, a weaker condition will specify the concept of gauge invariance in the quantum field theory of the electromagnetism as I will show in the sequel.

Let us now set up the Hilbert space of the physical states. To this concern, I will first define the Fock space $\mathcal{F}$ in the conventional way starting from the cyclic vacuum state

$$g_{\mathbf{k},A}\,|\,0\,\rangle = 0 = \langle\,0\,|\,g^{\dagger}_{\mathbf{k},A} \qquad \forall\,\mathbf{k}\in\mathbb{R}^3\, ,\ A = 1,2,L,S \tag{5.87}$$

a generic polarized $N-$photon energy momentum eigenstate being given by

$$|\,\mathbf{k}_1\,A_1\,\mathbf{k}_2\,A_2\,\ldots\,\mathbf{k}_N\,A_N\,\rangle \stackrel{\mathrm{def}}{=} \prod_{\jmath=1}^{N}\, g^{\dagger}_{\mathbf{k}_\jmath\,,A_\jmath}\,|\,0\,\rangle$$

It is very important to realize that the Fock space $\mathcal{F}$ for the massless gauge vector particles is of an indefinite metric. As a matter of fact, the inner product 4×4 real symmetric matrix $\eta \equiv \|\,\eta\,\|_{AB}$ $\;(\,A,B = 1,2,L,S\,)$ does satisfy

$$\eta^2 = \mathbb{I} \qquad\qquad \mathrm{tr}\,\eta = 2$$

which means that it admits three positive eigenvalues equal to $+1$ and one negative eigenvalue equal to -1 . Hence, negative norm states do indeed exist, for example

$$\frac{1}{\sqrt{2}}\left(g^{\dagger}_{\mathbf{k},L} + g^{\dagger}_{\mathbf{k},S} \right)\,|\,0\,\rangle$$

as well as null norm states just like

$$g^{\dagger}_{\mathbf{k},L}\,|\,0\,\rangle \qquad\qquad g^{\dagger}_{\mathbf{k},S}\,|\,0\,\rangle$$

Actually we readily find

$$\tfrac{1}{2}\langle\,0\,|\;(\,g_{\,\mathbf{h},L}+g_{\,\mathbf{h},S})\,(\,g^{\dagger}_{\mathbf{k},L}+g^{\dagger}_{\mathbf{k},S})\,|\,0\,\rangle \;=\; -\,\delta\,(\mathbf{h}-\mathbf{k}) \tag{5.88}$$

Then, an arbitrary *physical state* $|\,\mathrm{phys}\,\rangle\in\mathcal{H}_{\,\mathrm{phys}}\subset\mathcal{F}$ will be defined by the *auxiliary condition*

$$B^{\,(-)}\,(x)\,|\,\mathrm{phys}\,\rangle \;=\; 0 \tag{5.89}$$

where the positive frequency part of the auxiliary $B-$field is given by the normal mode expansion (5.54)

$$B^{\,(-)}\,(x) \;=\; \sum_{\mathbf{k}}\;\mathrm{i}\,k\cdot u_{\,\mathbf{k},S}\,(x)\,g_{\,\mathbf{k},S} \qquad\qquad (\,k_0=|\,\mathbf{k}\,|\,) \tag{5.90}$$

To understand the meaning of the auxiliary condition (5.89) consider the polarized 1-photon energy momentum eigenstates

$$|\,\mathbf{k}\,A\,\rangle = g^{\dagger}_{\mathbf{k},A}\,|\,0\,\rangle \qquad\qquad \langle\,B\,\mathbf{h}\,|\,\mathbf{k}\,A\,\rangle = \eta_{AB}\,\delta\,(\,\mathbf{h}-\mathbf{k}\,)$$

From the canonical commutation relations (5.76) it follows that the 1-photon states with transverse polarizations are physical

$$B^{\,(-)}\,(x)\,|\,\mathbf{k}\,A\,\rangle \;=\; 0 \qquad\qquad \forall\,\mathbf{k}\in\mathbb{R}^3 \qquad \forall\,A=1,2$$

as well as the scalar photon 1-particle states

$$B^{\,(-)}\,(x)\,|\,\mathbf{k}\,S\,\rangle \;=\; 0 \qquad\qquad \forall\,\mathbf{k}\in\mathbb{R}^3$$

Notice, however, that for any given wave packet $\varphi\,(\mathbf{k})$ normalized to one, *i.e.*

$$\int \mathrm{d}\mathbf{k}\;\;|\,\varphi\,(\mathbf{k})\,|^2 = 1$$

we find for $A,B=1,2$

$$\begin{aligned}
\langle\,\varphi_{\,B}\,|\,\varphi_{\,A}\,\rangle &= \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^*\,(\mathbf{k})\;\langle\,B\,\mathbf{k}\,|\,\mathbf{h}\,A\,\rangle = \delta_{\,AB}\\
\langle\,\varphi_{\,S}\,|\,\varphi_{\,S}\,\rangle &= \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^*\,(\mathbf{k})\;\langle\,S\,\mathbf{k}\,|\,\mathbf{h}\,S\,\rangle = 0\\
\langle\,\varphi_{\,S}\,|\,\varphi_{\,A}\,\rangle &= \int \mathrm{d}\mathbf{k}\int d\mathbf{h}\;\varphi\,(\mathbf{h})\,\varphi^*\,(\mathbf{k})\;\langle\,S\,\mathbf{k}\,|\,\mathbf{h}\,A\,\rangle = 0
\end{aligned}$$

From the above table of scalar products it follows that the 1-photon states with transverse polarizations are physical states with positive norm and are always orthogonal to the 1-particle scalar states, which are the physical states with zero norm. Hence, the 1-particle physical Hilbert space

$$\mathcal{H}_{1\,,\,\mathrm{phys}} \stackrel{\mathrm{def}}{=} \overline{V}_1 \qquad V_1 \equiv \left\{ \,|\,\mathbf{k}\,A\,\rangle\,|\,\mathbf{k}\in\mathbb{R}^3\ A=1,2,S\,\right\}$$

is a Hilbert space with a positive semidefinite metric. It is clear that the very same construction can be generalized in a straightforward way to define the N-particle completely symmetric physical Hilbert space – the closure of the symmetric product of 1–particle physical Hilbert spaces

$$\mathcal{H}_{N\,,\,\mathrm{phys}} \equiv \overline{\mathrm{V}}_N \qquad \mathrm{V}_N = \{\underbrace{\mathcal{H}_{1\,,\,\mathrm{phys}} \stackrel{\mathrm{s}}{\otimes} \mathcal{H}_{1\,,\,\mathrm{phys}} \stackrel{\mathrm{s}}{\otimes} \ldots \stackrel{\mathrm{s}}{\otimes} \mathcal{H}_{1\,,\,\mathrm{phys}}}_{\mathrm{N\ times}}\} = \mathcal{H}_{1\,,\,\mathrm{phys}}^{\stackrel{\mathrm{s}}{\otimes} n}$$

so that

$$\mathcal{H}_{\mathrm{phys}} \equiv \mathbf{C} \oplus \mathcal{H}_{1\,,\,\mathrm{phys}} \oplus \mathcal{H}_{2\,,\,\mathrm{phys}} \oplus \ \ldots\ \oplus \mathcal{H}_{N\,,\,\mathrm{phys}} \oplus \ \ldots\ = \bigoplus_{n=1}^{\infty} \mathcal{H}_{1\,,\,\mathrm{phys}}^{\stackrel{\mathrm{s}}{\otimes} n}$$

By their very construction, we see that the covariant physical photon states are equivalence classes of positive norm photon states with only transverse polarizations, up to the addition of any number of zero norm scalar photons.

This fact represents the quantum mechanical counterpart of the classical gauge transformations of the second kind. As a matter of fact, in classical electrodynamics the invariant Lorentz gauge condition $\partial \cdot A\,(x) = 0$ does not fix univocally the gauge potential, for a gauge transformation $A'_\mu\,(x) = A_\mu\,(x) + \partial_\mu\, f(x)$ with $f(x)$ satisfying the d'Alembert wave equation, is still compatible with the Lorentz condition. Hence, an equivalence class of gauge potentials obeying the invariant Lorentz condition indeed exists, what is known as classical gauge invariance of the second kind. Notice that such an invariance is no longer there for a non-covariant gauge condition, *e.g.* the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0\,$.

As a final remark, I'd like to stress that the notion of *gauge invariant local observable* in the covariant quantum theory of the free radiation field is as follows : a gauge invariant local observable $\mathfrak{O}\,(x)$ is a self-adjoint operator that maps the physical Hilbert space onto itself, *i.e.*

$$\mathfrak{O}\,(x)\,|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\mathrm{phys}} \qquad \forall\,|\,\mathrm{phys}\,\rangle \in \mathcal{H}_{\mathrm{phys}} \qquad \mathfrak{O}\,(x) = \mathfrak{O}^\dagger\,(x) \tag{5.91}$$

which implies

$$| \, \mathrm{phys} \, \rangle \in \mathcal{H}_{\mathrm{phys}} \quad \Leftrightarrow \quad B^{(-)}(x) \, | \, \mathrm{phys} \, \rangle = 0$$

$$\begin{aligned} B^{(-)}(x) \, \mathfrak{O}(y) \, | \, \mathrm{phys} \, \rangle &= \left[B^{(-)}(x) \, , \, \mathfrak{O}(y) \right] \, | \, \mathrm{phys} \, \rangle \\ &\propto \; B^{(-)}(x) \, | \, \mathrm{phys} \, \rangle = 0 \end{aligned}$$

It follows therefrom that the Maxwell field equation as well as the usual form of the energy momentum tensor for the radiation field hold true solely in a weak sense, *i.e.* as matrix elements between physical states : namely

$$\begin{aligned} & \langle \, \mathrm{phys}' \, | \, \partial_\mu F^{\mu\nu}(x) + \partial^\nu B(x) \, | \, \mathrm{phys} \, \rangle \; = \\ & \langle \, \mathrm{phys}' \, | \, \partial_\mu F^{\mu\nu}(x) \, | \, \mathrm{phys} \, \rangle = 0 \\ & \langle \, \mathrm{phys}' \, | \, \Theta_{\mu\nu}(x) \, | \, \mathrm{phys} \, \rangle \; = \\ & \langle \, \mathrm{phys}' \, | \, \tfrac{1}{4} g_{\mu\nu} \, F^{\rho\sigma}(x) \, F_{\rho\sigma}(x) - g^{\rho\sigma} \, F_{\mu\rho}(x) \, F_{\nu\sigma}(x) \, | \, \mathrm{phys} \, \rangle \\ & \forall \; | \, \mathrm{phys} \, \rangle \, , \, | \, \mathrm{phys}' \rangle \; \in \mathcal{H}_{\mathrm{phys}} \end{aligned}$$

In respect to the above definition, the canonical energy momentum tensor of the massless vector gauge field theory is neither symmetric nor observable, as it appears to be evident from its expression

$$\begin{aligned} T_{\mu\nu} &= \; : A_\mu \, \partial_\nu B - \; F_{\mu\lambda} \, \partial_\nu \, A^\lambda \\ &+ \; g_{\mu\nu} \left(\tfrac{1}{4} F^{\rho\sigma} F_{\rho\sigma} - A^\lambda \, \partial_\lambda B - \tfrac{1}{2} \, \xi \, B^2 \right) \, : \end{aligned} \tag{5.92}$$

because, from the canonical commutation relations (5.79), we immediately get

$$\begin{aligned} [\, B(x) \, , \, T_{\mu\nu}(y) \,] &= \; - \mathrm{i} \, \partial_{\mu\,,\,x} \, \triangle(x-y) \, \partial_\nu \, B(y) \\ &+ \; \mathrm{i} \, g_{\mu\nu} \, \partial_x^\lambda \, \triangle(x-y) \, \partial_\lambda B(y) \\ &- \; \mathrm{i} \, F_{\mu\lambda}(y) \, \partial_{\nu\,,\,x} \, \partial_x^\lambda \, \triangle(x-y) \end{aligned}$$

which does not fulfill the criterion (5.91) owing to the presence of the very last term. Conversely, the symmetric energy momentum tensor

$$\begin{aligned} \Theta_{\mu\nu} &\stackrel{\mathrm{def}}{=} \; : A_\mu \, \partial_\nu B + A_\nu \, \partial_\mu B - g^{\lambda\rho} \, F_{\mu\lambda} \, F_{\nu\rho} - g_{\mu\nu} \, \mathcal{L}_{A\,,\,B} \, : \\ &= \; : A_\mu \, \partial_\nu B + A_\nu \, \partial_\mu B - g^{\lambda\rho} \, F_{\mu\lambda} \, F_{\nu\rho} \\ &+ \; g_{\mu\nu} \left(\tfrac{1}{4} F^{\rho\sigma} F_{\rho\sigma} - A^\lambda \, \partial_\lambda B - \tfrac{1}{2} \, \xi \, B^2 \right) \, : \end{aligned}$$

as well as, *a fortiori*, the energy momentum four vector do indeed satisfy the requirements (5.91), *i.e.* they are observable in the quantum mechanical sense, since we have

$$\begin{aligned} [\, B(x)\,,\, \Theta_{\mu\nu}(y)\,] &= -\,\mathrm{i}\,\partial_{\mu\,,\,x}\,\triangle\,(x-y)\,\partial_{\nu}\,B\,(y) \\ &- \;\mathrm{i}\,\partial_{\nu\,,\,x}\,\triangle\,(x-y)\,\partial_{\mu}\,B\,(y) \\ &+ \;\mathrm{i}\,g_{\mu\nu}\,\partial_{x}^{\lambda}\,\triangle\,(x-y)\,\partial_{\lambda}\,B\,(y) \end{aligned}$$

$$\Theta_{\mu\nu}(y) = \Theta_{\nu\mu}(y) \qquad\qquad \Theta_{\mu\nu}(y) = \Theta^{\dagger}_{\mu\nu}(y)$$

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5.4 Problems

1. **Covariance of the vector field.** Find the transformation laws of the quantized vector wave field under the Poincaré group.

   Solution. We shall treat in detail the massless case, the generalization to the massive case being straightforward. We have to recall some standard definitions : namely,

$$\int \triangle k \overset{\text{def}}{=} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\, 2\,|\,\mathbf{k}\,|} = \frac{1}{(2\pi)^3} \int \mathrm{d}k\ \theta(k_0)\,\delta\left(k^2\right) \tag{5.93}$$

$$|\,k\,A\,\rangle \overset{\text{def}}{=} [\,(2\pi)^3\, 2\,|\,\mathbf{k}\,|\,]^{\frac{1}{2}}\, g^{\dagger}_{\mathbf{k}\,,\,A}\,|\,0\,\rangle = g^{\dagger}_{A}\,(k)\,|\,0\,\rangle \tag{5.94}$$

covariant 1-particle states for the massless vector field

$$\left\{\,|\,k\,A\,\rangle = [\,(2\pi)^3\, 2\,|\,\mathbf{k}\,|\,]^{\frac{1}{2}}\, g^{\dagger}_{\mathbf{k}\,,\,A}\,|\,0\,\rangle\;|\;\; \mathbf{k} \in \mathbb{R}^3\,\right\} \tag{5.95}$$

orthonormality and closure relations

$$\langle\, h\,A\,|\,k\,B\,\rangle = (2\pi)^3\, 2\,|\,\mathbf{k}\,|\,\delta\,(\mathbf{h}-\mathbf{k})\,\eta_{AB}$$
$$\sum_A \int \triangle k\ |\,k\,A\,\rangle\,\langle\,k\,A\,| = \mathbf{I}_1$$

gauge potential covariant normal mode expansion

$$A^{\lambda}\,(x) = \sum_A \int \triangle k\ \left[\,\varepsilon^{\lambda}_{A}\,(k)\,g_{A}\,(k)\,e^{-ikx} + \varepsilon^{\lambda *}_{A}\,(k)\,g^{\dagger}_{A}\,(k)\,e^{ikx}\,\right]$$

wave functions

$$u^{\lambda}_{k}\,(x) \equiv \langle\,0\,|\,A^{\lambda}\,(x)\,|\,k\,A\,\rangle = \varepsilon^{\lambda}_{A}\,(k)\ \exp\{-i\,k\cdot x\} \qquad (\,k_0 = |\,\mathbf{k}\,|\,)$$
$$\int d\mathbf{x}\ u^{\nu *}_{h}\,(x)\,i\,\overleftrightarrow{\partial}_0\,u^{\lambda}_{k}\,(x) = 2\,|\,\mathbf{k}\,|\,\delta\,(\,\mathbf{h}-\mathbf{k}\,)$$

For each element of the restricted Lorentz group, which is univocally specified by the six canonical coordinates $\omega^{\mu\nu} = (\boldsymbol{\alpha},\boldsymbol{\beta})$, there will correspond a *unitary operator* $U(\omega)\ :\ \mathcal{F}_1 \rightarrow \mathcal{F}_1$ so that

$$\begin{aligned}
U(\omega)\,|\,k\,A\,\rangle &= |\,\Lambda\,k\ A\,\rangle \\
\langle\,A\,k\,|\,U^{\dagger}(\omega) &= \langle\,A\ \Lambda\,k\,| \\
\langle\,A\ \Lambda\,h\,|\,\Lambda\,k\ B\,\rangle &= \langle\,A\,h\,|\,k\,B\,\rangle \\
&= \langle\,A\,h\,|\,U^{\dagger}(\omega)\,U(\omega)\,k\,B\,\rangle \\
&= \eta_{AB}\ \delta\,(\,\mathbf{h}-\mathbf{k}\,)\ (2\pi)^3\, 2\,|\,\mathbf{k}\,|
\end{aligned}$$

where we obviously understand *e.g.*

$$\begin{aligned}
&|\,A\ \Lambda\, k\,\rangle = |\,A\, k'\,\rangle = [\,(2\pi)^3\, 2\, k_0'\,]^{\frac{1}{2}}\, g^{\dagger}_{\mathbf{k}',\,A}\,|\,0\,\rangle = g^{\dagger}_A\,(\Lambda\, k)\,|\,0\,\rangle \\
&k'_\mu = \Lambda_\mu^{\ \nu}\, k_\nu \qquad k_0' = |\,\mathbf{k}'\,| \\
&g^{\mu\nu}\, k'_\mu\, k'_\nu = k^2 = 0
\end{aligned}$$

Under a Lorentz transformation, the creation annihilation operators do transform according to the law

$$\begin{aligned}
U(\omega)\, g^{\dagger}_A\,(k)\, U^{\dagger}(\omega) &= g^{\dagger}_A\,(\Lambda\, k) \\
U(\omega)\, g_A\,(k)\, U^{\dagger}(\omega) &= g_A\,(\Lambda\, k)
\end{aligned}$$

under the assumption that the vacuum is invariant *i.e.* $U(\omega)\,|\,0\rangle = |\,0\,\rangle$. As a consequence

$$\begin{aligned}
A'^{\lambda}\,(x) &\stackrel{\text{def}}{=} U(\omega)\, A^{\lambda}\,(x)\, U^{\dagger}(\omega) \\
&= \sum_A \int \triangle k\, \left[\, \varepsilon^{\lambda}_A\,(k)\, g_A\,(\Lambda\, k)\, e^{-ikx} + \varepsilon^{\lambda *}_A\,(k)\, g^{\dagger}_A\,(\Lambda\, k)\, e^{ikx} \right] \\
&= \Lambda^{\lambda}_{\ \ \sigma} \sum_A \int \triangle k'\, \left[\, \varepsilon'^{\sigma}_A\,(k')\, g_A\,(k')\, \exp\{-i\, k'\cdot x'\} \right. \\
&+ \left. \varepsilon'^{\sigma *}_A\,(k')\, g^{\dagger}_A\,(k')\, \exp\{\, i\, k'\cdot x'\} \right] \\
&= \Lambda^{\lambda}_{\ \ \sigma}\, A^{\sigma}\,(\Lambda\, x)
\end{aligned}$$

in which I have used the standard four vector transformation rule (2.21)

$$\varepsilon^{\lambda}_A\,(k) = \Lambda^{\lambda}_{\ \ \sigma}\, \varepsilon'^{\sigma}_A\,(k')$$

the new equivalent basis of the polarization vectors $\{\, \varepsilon'^{\mu}_A\,(k'\,)\,\}$ still obeying the orthonormality and closure relations

$$-\, g_{\mu\nu}\, \varepsilon'^{\mu *}_A\,(k')\, \varepsilon'^{\nu}_B\,(k') = \eta_{AB}$$

$$\eta_{AB}\, \varepsilon'^{\mu *}_A\,(k')\, \varepsilon'^{\nu}_B\,(k') = -\, g^{\,\mu\nu}$$

2. **The generating functional.** Construct the generating functional for the massive and massless real vector free field theories.

   Solution. This can be done by a straightforward generalization of the real scalar field case. In particular, we can conveniently define

$$u^A\,(x) \stackrel{\text{def}}{=} (\,A^{\mu}\,(x)\,,\, B\,(x)\,) \qquad A = (\,\mu\,,\,\bullet\,)$$

$$j_A(x) \stackrel{\text{def}}{=} (J_\mu(x),\, K(x))$$

$$\int \mathrm{d}x\ u^A(x)\, j_A(x) = \int \mathrm{d}x\ \Big[A^\mu(x)\, J_\mu(x) + B(x)\, K(x) \Big]$$

so that we have a 5-dimensional diagonal metric tensor

$$g^{AB} = g_{AB} = \mathrm{diag}(+1, -1, -1, -1, +1)$$

The classical action can be written in the form

$$S_0[u] \ = \ -\frac{1}{2} \int \mathrm{d}x\ u^A(x)\, K^{(x)}_{AB}\, u^B(x)$$

with

$$K^{(x)}_{\mu\nu} = -\,g_{\mu\nu}\left(\Box + m^2\right) + \partial_\mu \partial_\nu \qquad K^{(x)}_{\mu\bullet} = \partial_\mu \qquad K^{(x)}_{\bullet\bullet} = \xi$$

The kinetic operator $K^{(x)}_{AB}$ can be univocally inverted by means of the causal $(+\,i\varepsilon)-$prescription leading to the causal Green's function

$$D^{(c)}_{AB}(x - y\,;\, m\,,\, \xi) = \int \frac{\mathrm{d}k}{(2\pi)^4}\ \widetilde{D}^{(c)}_{AB}(k\,;\, m\,,\, \xi)\ \exp\{-\,i\,k\cdot(x-y)\}$$

$$\widetilde{D}^{F}_{\mu\nu}(k\,;\, m\,,\, \xi) = \frac{i}{k^2 - m^2 + i\,\varepsilon}\left\{-\,g_{\mu\nu} + \frac{(1-\xi)\,k_\mu\, k_\nu}{k^2 - \xi\, m^2 + i\,\varepsilon'}\right\}$$

$$\widetilde{D}^{F}_{\nu\bullet}(k\,;\, m\,,\, \xi) = \frac{k_\nu}{k^2 - \xi\, m^2 + i\,\varepsilon'}$$

$$\widetilde{D}^{F}_{\bullet\bullet}(k\,;\, m\,,\, \xi) = \frac{-\,i\,m^2}{k^2 - \xi\, m^2 + i\,\varepsilon'}$$

In fact

$$g^{\nu\rho}\, K^{(x)}_{\mu\nu}\, D^F_{\rho\sigma}(x) + K^{(x)}_{\mu\bullet}\, D^F_{\sigma\bullet}(x) = -\,i\, g_{\mu\sigma}\, \delta(x)$$
$$g^{\mu\nu}\, K^{(x)}_{\mu\bullet}\, D^F_{\nu\bullet}(x) + \xi\, D^F_{\bullet\bullet}(x) = -\,i\, \delta(x)$$

This means that we can write

$$g^{BC}\, K^{(x)}_{AB}\, D^{(c)}_{CD}(x) = -\,i\, g_{AD}\, \delta(x)$$

and thereby

$$\begin{aligned} Z_0[j] \ &= \ \left\langle T\ \exp\left\{ \mathrm{i} \int \mathrm{d}x\ u_A(x)\, j^A(x) \right\} \right\rangle_0 \\ &= \ \exp\left\{ -\tfrac{1}{2} \textstyle\int \mathrm{d}x \int \mathrm{d}y\ D^{(c)}_{AB}(x-y)\, j^A(x)\, j^B(y) \right\} \\ &= \ \mathcal{N} \int \mathfrak{D}\, u_A\ \exp\left\{ \mathrm{i}\, S_0[u] + \mathrm{i} \int \mathrm{d}x\ u^A(x)\, j_A(x) \right\} \end{aligned}$$

with

$$Z_0[0] = \mathcal{N} \int \mathfrak{D}\, u_A \;\exp\{i\, S_0[u]\} = 1$$

Turning to the euclidean formulation in the usual way

$$x_4 = i\, x_0 \qquad\qquad A_4(x_E) = i\, A_0(-\,i\, x_4, \mathbf{x})$$

$$-\, g_{\mu\nu}\, x^\mu x^\nu = x_E^2 \qquad\qquad -\; g_{\mu\nu}\, A^\mu(x)\, A^\nu(x) = A_{E\mu}(x_E)\, A_{E\mu}(x_E)$$

et cetera, we find

$$Z_E^{(0)}[0] = \mathcal{N} \int \mathfrak{D}\, A_{E\mu} \int \mathfrak{D}\, B_E \;\exp\left\{-\, S_E^{(0)}[A_{E\mu}\,,\, B_E]\right\} = 1$$

where

$$\begin{aligned} S_E^{(0)}[A_{E\mu}\,,\, B_E] &= \tfrac{1}{2} A_{E\mu}\left[(m^2 - \partial_E^2)\,\delta_{\mu\nu} + \partial_{E\mu}\partial_{E\nu}\right] A_{E\nu} \\ &+ \; A_{E\mu}\,\partial_{E\mu} B_E - \tfrac{1}{2}\,\xi\, B_E^2 \end{aligned}$$

After rescaling with an arbitrary mass scale μ

$$A_{E\mu} \;\mapsto\; A'_{E\mu} = \mu\, A_{E\mu} \qquad\qquad B_E \;\mapsto\; B'_E = \mu^2\, B_E$$

we come to

$$Z_E^{(0)}[0] = \mathcal{N}'\, \det \parallel K_{\mathrm{E}} \parallel$$

$$K_{\mathrm{E}} \;=\; \mu^{-2} \begin{pmatrix} (m^2 - \partial_\lambda\partial_\lambda)\,\delta_{\mu\nu} + \partial_\mu\,\partial_\nu & \partial_\rho \\ \partial_\rho & -\,\xi\,\mu^2 \end{pmatrix}$$

Hence, from the $\zeta-$function regularization technique we obtain for $\Re\mathrm{e}\,\xi\,\mu^2 < 4m^2$

$$\begin{aligned} \mathrm{Tr}\, K_{\mathrm{E}}^{-s} &= V\,\mu^{2s} \int \frac{\mathrm{d}k}{(2\pi)^4}\,\left(4m^2 + 3k^2 - \xi\,\mu^2\right)^{-s} \\ &= \frac{V\,\mu^{2s}}{144\pi^2\,\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-3}\,\exp\left\{-t\left(4m^2 - \xi\,\mu^2\right)\right\} \\ &= \frac{V\, m^2}{9\pi^2}\left(1 - \frac{\xi\,\mu^2}{4m^2}\right)^2\left(\frac{4m^2}{\mu^2} - \xi\right)^{-s}\frac{\Gamma(s-2)}{\Gamma(s)} \\ &= \frac{V\, m^2}{9\pi^2}\left(1 - \frac{\xi\,\mu^2}{4m^2}\right)^2\left(\frac{4m^2}{\mu^2} - \xi\right)^{-s}(s^2 - 3s + 2)^{-1} \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{ds}\,\mathrm{Tr}\, K_{\mathrm{E}}^{-s} &= (s^2 - 3s + 2)^{-1}\,\mathrm{Tr}\, K_{\mathrm{E}}^{-s} \\ &\times \;\left[3 - 2s - (s^2 - 3s + 2)\,\ln\left(\frac{4m^2}{\mu^2} - \xi\right)\right] \end{aligned}$$

and thereby

$$\det \parallel K_{\mathrm{E}} \parallel = \frac{V\,m^2}{9\pi^2}\left(1-\frac{\xi\,\mu^2}{4m^2}\right)^2\left[\frac{1}{2}\,\ln\left(\frac{4m^2}{\mu^2}-\xi\right)-\frac{3}{4}\right]$$

so that the normalization condition $Z_E^{(0)}[\,0\,]=1$ yields

$$\mathcal{N}' \equiv \exp\left\{\frac{V\,m^2}{18\pi^2}\left(1-\frac{\xi\,\mu^2}{4m^2}\right)^2\left[\frac{1}{2}\,\ln\left(\frac{4m^2}{\mu^2}-\xi\right)-\frac{3}{4}\right]\right\}$$

The transition to the Minkowski spacetime can be immediately done by simply replacing $V_{\text{euclidean}} \quad\leftrightarrow\quad i\,V_{\text{minkowskian}}$ which leads to the final result

$$\begin{aligned}
Z_0[\,j\,] &= \left\langle T\,\exp\left\{\mathrm{i}\int \mathrm{d}x\; u_A(x)\,j^{\,A}(x)\right\}\right\rangle_0 \\
&= \exp\left\{-\tfrac{1}{2}\textstyle\int \mathrm{d}x \int \mathrm{d}y\; D^{(c)}_{AB}(x-y)\,j^{\,A}(x)\,j^{\,B}(y)\right\} \\
&= \mathcal{N}\int \mathfrak{D}\,u_A\;\exp\left\{\mathrm{i}\,S_0[\,u\,]+\mathrm{i}\int \mathrm{d}x\; u^A(x)\,j_A(x)\right\} \\
\mathcal{N} &= \text{constant}\times\left(\det K^{(x)}_{AB}\right)^{1/2} \\
&\stackrel{\text{def}}{=} \exp\left\{\frac{\mathrm{i}V\,m^2}{18\pi^2}\left(1-\frac{\xi\,\mu^2}{4m^2}\right)^2\left[\frac{1}{2}\,\ln\left(\frac{4m^2}{\mu^2}-\xi\right)-\frac{3}{4}\right]\right\}
\end{aligned}$$

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