09520 - TEORIA DEI CAMPI (Theory of Fields)

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Assessment tests

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Contents

Problems

0.1 Thursday 10 September 2009

FIELD THEORY **1.**

The minimal Goldstone model : consider the classical Lagrangian for a real scalar field

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu \phi(x)\,\partial^\mu \phi(x) + \tfrac{1}{2}\,\mu^2\,\phi^2(x) - \tfrac{1}{4}\,\lambda\phi^4(x)$$

with $\mu \in \mathbb{R}$ and $\lambda > 0$

1. Find all the symmetries of the above field theoretic model
2. Write the energy-momentum tensor as well as the total energy and total momentum for this model and comment about it
3. Determine all the degenerate classical field configurations of minimal energy and specify their symmetry
4. Choose one of the degenerate minimum field configurations $\phi_{\min}$ for the classical Hamiltonian of the minimal Goldstone model and expand the Lagrangian around it : which are the symmetries of the resulting new classical Action? Comment about the classical Hamiltonian as a function of the shifted field $\varphi = \phi - \phi_{\min}$

FIELD THEORY

Consider the Compton scattering $e^-\gamma \;\to\; e^-\gamma$. The initial state contains a photon of momentum $\mathbf{k}$, energy $k = |\,\mathbf{k}\,|$, and an electron at rest of mass m_e. In the final state we have the scattered photon of momentum $\mathbf{k}'$, energy $k' = |\,\mathbf{k}'\,|$, and an electron which has received the recoil momentum $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$ and with the energy

$$p_0' = E' = \sqrt{(\mathbf{k} - \mathbf{k}')^2 + m_e^2} = \sqrt{\mathbf{k}^2 + \mathbf{k}'^2 - 2\mathbf{k}\cdot\mathbf{k}'\,\cos\theta + m_e^2}$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$. The electron spin states are u_r and $\bar{u}_{r'}(\mathbf{p}')$ while the photon linear polarization vectors are $\varepsilon_\mu^{A'}(k')$ and $\varepsilon_\nu^{A}(k)$ respectively, with $r, r', A, A' = 1, 2$.

1. Write the probability amplitude $i\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}')$ for this process to the lowest order in the fine structure constant
2. Find the square modulus of the probability amplitude averaging over the initial and summing over the final electron polarizations

0.2 Thursday 9 July 2009

FIELD THEORY **1.**

Consider the classical Lagrangian for the real scalar field

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi(x)\,\partial^\mu \phi(x) - \frac{1}{2}\,m^2\,\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x)$$

and the dilatation trasformation abelian Lie group which acts on space-time coordinates and field according to

$$x^\mu \;\to\; y^\mu = \mathrm{e}^{\alpha}\,x^\mu \qquad\qquad \phi(x) \;\to\; \phi'(y) = \mathrm{e}^{-\alpha}\,\phi(x)$$

1. Calculate the canonical energy-momentum symmetric tensor $T^{\mu\nu}$
2. Calculate the Noether current $J^\mu(x)$ associated to the Lie group of the dilatation transformations and discuss its continuity equation
3. Define the new *improved* energy-momentum symmetric tensor

   $$\Theta_{\mu\nu} = T_{\mu\nu} + a\,(\,\partial_\mu\,\partial_\nu - g_{\mu\nu}\,\Box\,)\;\phi^2(x) \qquad\qquad (\,a \in \mathbb{R}\,)$$

   and determine the constant a in such a manner that

   $$g^{\mu\nu}\,\Theta_{\mu\nu} = 0 \qquad\qquad \text{for } m = 0 \quad \text{(traceless condition)}$$

4. Define a new dilatation current

   $$j_\mu(x) \equiv x^\rho\,\Theta_{\mu\rho}$$

   and show that

   $$\partial_\mu\,j^\mu(x) = g^{\lambda\nu}\,\Theta_{\lambda\nu}(x)$$

FIELD THEORY

Evaluate the differential cross section to the lowest order $O(\alpha^2)$, where α is the fine structure constant, for the electron-proton collision $e^-\,p \;\longrightarrow\; e^-\,p$ in the proton rest frame, disregarding the electron mass in respect to the proton mass and the proton recoil after the collision. (*Nota Bene* : the proton is assumed to be point-like)

0.3 Friday 12 June 2009

FIELD THEORY **1.**

1. Given the spinor Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\, i\partial\!\!\!/\, \psi(x) - M\, \psi(x)\, \psi(x) - i\, M'\, \bar{\psi}(x)\, \gamma_5\, \psi(x)$$

where ψ is a complex four components Dirac bispinor, whereas M and M' are positive mass parameters, use the chiral phase transformation

$$\psi(x) \longrightarrow \psi'(x) = \exp\{-\, i\alpha\, \gamma_5\}\, \psi(x) \qquad (\, 0 \le \alpha < 2\pi\,)$$

to remove the pseudoscalar mass term. What is the mass of the resultant Dirac field?

2. Write the Poincaré invariant classical Lagrangian for the radiation field, *i.e.* the free electromagnetic field, in the Feynman gauge.
Write the Euler-Lagrange field equations and find the corresponding plane wave solution for a given energy-momentum and polarization.
What is the plane wave solution which describes a physical photon of given momentum $\mathbf{k}$ and transverse polarization?

FIELD THEORY

1. Given the spinor Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\, i\partial\!\!\!/\, \psi(x) - M\, \bar{\psi}(x)\, \psi(x) - i\, M'\, \bar{\psi}(x)\, \gamma_5\, \psi(x)$$

where ψ is a complex four components Dirac bispinor, whereas M and M' are positive mass parameters, use a chiral phase transformation to remove the pseudoscalar mass term. What is the mass of the resultant Dirac field?

2. Calculate the divergent one−loop Feynman integral

$$\Sigma_1\,(\emptyset) \stackrel{\text{def}}{=} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \mathrm{tr}\, \frac{-\,i\,g}{p\!\!\!/ - M + i\varepsilon} \qquad \text{(spinor tadpole)}$$

using two different regularisation techniques

1. large ultra−violet wave number K cut-off regularisation,
2. dimensional regularisation.

Discuss the results.

0.4 Friday 29 May 2009

FIELD THEORY **1.**

Consider the planar Chern-Simons spinor electrodynamics, which is the field theory defined on the *2+1* dimensional Minkowski space-time with

$$x^{\mu} = (x^0, x^1, x^2) = (ct, x, y)$$

metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and described by the classical the Lagrangian density

$$\mathcal{L} = \frac{1}{4}\,\varkappa\,\varepsilon^{\lambda\mu\nu}\,A_{\lambda}(x)\,F_{\mu\nu}(x) + \bar{\psi}(x)\,[\,\mathrm{i}\not\partial + q\,\not A(x) - M\,]\,\psi(x)$$

where $\not a \equiv \gamma^{\mu}\,a_{\mu}$ and $\varepsilon^{012} = 1$, which involves the real vector potential $A_{\lambda}(x)$, the field strength $F_{\mu\nu}(x) = \partial_{\mu}\,A_{\nu}(x) - \partial_{\nu}\,A_{\mu}(x)$, the two component spinor

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

q being the charge of spinor particle, $\varkappa$ a real number known as the Chern-Simoms coefficient and, finally, the 2×2 gamma matrices

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = \mathrm{i}\,\sigma_2 \qquad \gamma^2 = \mathrm{i}\,\sigma_3$$

where $\sigma_{\imath}$ ($\imath = 1, 2, 3$) are the Pauli matrices, in such a manner that the Clifford algebra $\{\gamma^{\mu}\,,\,\gamma^{\nu}\} = 2g^{\mu\nu}\,\mathbb{I}$ holds true.

1. Determine the canonical dimensions of the gauge potential and spinor field in natural units as well as in physical units
2. Find the Euler-Lagrange field equations and make some comments
3. Specify the conditions under which the Action is gauge invariant
4. Find the normal mode decomposition of the quantized free spinor field, *i.e.* for $q = 0$, in the massless case $M = 0$

Score : 1. pt. 2, 2. pt. 3, 3. pt. 10 ; 4. pt 15

FIELD THEORY

Consider the planar Spinor Electrodynamics, *i.e.* , the field theory defined on the *2+1* dimensional Minkowski space-time with metric tensor diag$(+,-,-)$. The classical dynamics is governed by the Lagrangian density

$$\mathcal{L}_{c\ell} = \tfrac{1}{4}\,\varkappa\,\varepsilon^{\lambda\mu\nu} A_\lambda(x)\,F_{\mu\nu}(x) + \bar\psi(x)\,[\,\mathrm{i}\partial\!\!\!/ + q\,A\!\!\!/(x) - M\,]\,\psi(x)$$

where $a\!\!\!/ \equiv \gamma^\mu a_\mu$ and $\varepsilon^{012} = 1$, which involves the real vector potential $A_\lambda(x)$, the field strength $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and the spinor

$$\psi(x) = \left(\begin{array}{c}\psi_1(x)\\ \psi_2(x)\end{array}\right)$$

q being the charge of spinor particle, $\varkappa$ a real number and the $\gamma-$matrices

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = \mathrm{i}\sigma_2 \qquad \gamma^2 = \mathrm{i}\sigma_3$$

where $\sigma_\imath$ ($\imath = 1,2,3$) are the Pauli matrices, in such a manner that the Clifford algebra $\{\gamma^\mu\,,\,\gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{I}$ holds true.

1. Specify the Feynman rules in momentum space, which follow from the classical Lagrangian

2. Find the most general gauge and Poincaré invariant Action functional, at most quadratic in the gauge field strength and in the spinor field

3. Write the expression for the gauge particle self-energy, as it follows from the classical Lagrangian, and evaluate its degrees of divergence by naïve power counting

4. The latter can be evaluated using dimensional regularisation and yields

$$\begin{aligned}\mathtt{reg}\,\Pi^{\mu\nu}(k,M,\mu) &= \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\mathtt{reg}\,\Pi(k^2, M^2)\\ \mathtt{reg}\,\Pi(k^2, M^2) &= \frac{-4q^2}{(4\pi)^\omega}\int_0^1 \frac{x(1-x)\,\Gamma(2-\omega)\,\mathrm{d}x}{[\,M^2 - x(1-x)\,k^2\,]^{2-\omega}}\end{aligned}$$

   in 2ω space-time dimensions : calculate the above expression for $\omega = \frac{3}{2}$

5. Write the corresponding Lagrangian with $\Pi(0, M^2)$

Score : 1. pt. 2, 2. pt. 5, 3. pt. 3 ; 4. pt 15 ; 5. pt. 5

0.5 Thursday 22 January 2009

1. Consider two inertial reference frame K and K' with parallel axes and with coincident origins and clocks at $t = 0 = t'$, such that K' is moving with a velocity $\beta = v_z/c > 0$ along the positive $Oz-$axis
[*nota bene* : use $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$]

1. Write down the passive Lorentz transformation $\Lambda(\beta)$ relating the space-time coordinates of the two reference frames and belonging to the $\boldsymbol{\tau}_{\frac{1}{2}\frac{1}{2}}$ vector representation
2. Write the corresponding matrix $\Lambda_L(\beta)$ of the Weyl representation $\boldsymbol{\tau}_{\frac{1}{2}0}$
3. As well, find the corresponding matrix $\Lambda_{\frac{1}{2}}(\beta)$ of the representation $\boldsymbol{\tau}_D = \boldsymbol{\tau}_{\frac{1}{2}0} \oplus \boldsymbol{\tau}_{0\frac{1}{2}}$ using the Weyl (or spinor, chiral) representation for the $\gamma-$matrices

2. Consider a Klein-Gordon real quantum field $\phi(t, \mathbf{x})$ and the corresponding particle number operator

$$\mathrm{N} \equiv \int \mathrm{d}\mathbf{k}\; a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}}$$

Find the commutator $[\,\mathrm{N}\,,\, P_\mu\,]$, where P_μ is the energy-momentum operator

3. Consider a Dirac spinor quantum free field $\psi(t, \mathbf{x})$ and its normal mode decomposition

$$\psi(x) = \sum_{\mathbf{p}\,,\,r} \left[c_{\mathbf{p}\,,\,r}\, u_{\mathbf{p}\,,\,r}(x) + d^\dagger_{\mathbf{p}\,,\,r}\, v_{\mathbf{p}\,,\,r}(x) \right]$$

$$\bar\psi(x) = \sum_{\mathbf{p}\,,\,r} \left[c^\dagger_{\mathbf{p}\,,\,r}\, \bar u_{\mathbf{p}\,,\,r}(x) + d_{\mathbf{p}\,,\,r}\, \bar v_{\mathbf{p}\,,\,r}(x) \right]$$

1. Show that
$$\{\psi(x)\,,\,\bar\psi(y)\} = (i\,\partial\!\!\!/_x + M) \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\; \mathrm{e}^{-ip(x-y)}\; \delta(p^2 - M^2)\, \mathrm{sgn}(p_0)$$
2. Show that for $x^0 = y^0$ the above anticommutator becomes $\gamma^0\,\delta(\mathbf{x} - \mathbf{y})$

Score : **1.**1. pt. 3, **1.**2. pt. 3, **1.**3. pt. 3 ; **2.** pt 5 ; **3.**1. pt. 10, **3.**2. pt. 5

0.6 Wednesday 16 July 2008

1. Consider the classical Lagrange density for spinor electrodynamics

$$\mathcal{L} = \tfrac{1}{2}\,\bar{\psi}(x)\,\gamma^{\mu}\,\mathrm{i}\,\overleftrightarrow{\partial}_{\mu}\,\psi(x) - M\,\bar{\psi}(x)\,\psi(x) + eA_{\mu}(x)\,\bar{\psi}(x)\,\gamma^{\mu}\,\psi(x)$$

   and the *chiral* phase transformations $(\,0 \le \alpha < 2\pi\,)$

$$\psi'(x) = \exp\{-\,\mathrm{i}\,\alpha\,\gamma_5\}\,\psi(x) \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,\exp\{-\,\mathrm{i}\,\alpha\,\gamma_5\}$$

   Find the continuity equation satisfied by the classical *axial current*

$$J_5^{\mu}(x) = \bar{\psi}(x)\,\gamma^{\mu}\,\gamma_5\,\psi(x)$$

   and determine the conditions under which the axial charge is conserved.

2. The charge conjugation, parity and time reversal transformations on Dirac spinors are respectively provided by

$$\mathcal{C}\,\psi(t,\mathbf{x})\,\mathcal{C} = \psi^{c}(x) = -\,\mathrm{i}\,\gamma^{2}\,\left(\psi^{\dagger}(x)\right)^{\top} \;=\; \left(-\,\mathrm{i}\,\bar{\psi}(x)\,\gamma^{0}\,\gamma^{2}\right)^{\top}$$

$$\mathcal{P}\,\psi(t,\mathbf{x})\,\mathcal{P} = \gamma^{0}\,\psi(t,-\,\mathbf{x})$$

$$\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T}^{\dagger} = \Theta\,\psi(-\,t,\mathbf{x}) \qquad \Theta \;=\; -\,\gamma^{1}\,\gamma^{3}$$

   Find the transformation laws of the axial current under $\mathcal{C}\,\mathcal{P}\,\mathcal{T}$

0.7 Friday 6 June 2008

1. The non$-$abelian vector potential in the four dimensional Minkowski space$-$time is defined by

$$A_\mu(x) = A_\mu^a(x)\,\boldsymbol{\tau}_A^a(g) \qquad g \in G \;=\; SU(3)_c \otimes SU(2)_f \otimes U(1)_Y$$

where the hermitean matrices $\boldsymbol{\tau}_A^a(g)$ $(a = 1, 2, \ldots, n = \dim G)$ belong to the adjoint representation of the internal symmetry unitary group. How many real independent field components does it contain, if the vector field describe massive particles? How many in the case of a non$-$abelian gauge vector field?

A fundamental spinor matter field, describing *e.g.* the first family of up and down quark fields, is a massive Dirac field belonging to the fundamental representation $\boldsymbol{\tau}_F$ of the very same group G. How many real field components does it contain?

2. Consider the process $e^+e^- \;\rightarrow\; \mu^+\mu^-$:

(*i*) write down the exact invariant expression for the square of the probability amplitude to the lowest order in perturbation theory after averaging over initial and summing over final spin polarizations

(*ii*) find an approximate $O(\alpha^2)$ expression for the differential and the total cross sections in the center of momentum frame for

$$m_e \approx 0 \qquad m_\mu^2 \ll s$$

where s denotes the conventional Mandelstam variable.

(*iii*) The present day values of the top quark mass and charge are

$$\begin{aligned} m_t &= 174.2 \pm 3.3 \text{ GeV} && (\text{direct observation of top events}) \\ &= 172.3^{+10.2}_{-7.6} \text{ GeV} && (\text{Standard Model electroweak fit}) \\ \text{charge} &= e\,Q_t = \frac{2}{3}\,e \end{aligned}$$

Find the leading value for the lowest order total cross section

$$\sigma(\,e^+e^- \;\rightarrow\; \text{hadrons}\,)$$

when $s > 4m_t^2$ and in the ultrarelativistic limit.

[1 unit of R $\equiv\; 4\pi\alpha^2/3E_{CM}^2 = (\hbar c)^2$ 86.8 nbarns (E_{CM} in GeV)$^{-2}$]

Solutions

0.8 Thursday 10 September 2009

TEORIA DEI CAMPI **1.**

1. The invariance symmetry groups of the model are the inhomogeneous full Lorentz group IO(1,3), or full Poincaré group, and the internal discrete symmetry group $\mathbb{Z}_2$

2. The canonical energy momentum tensor of the model, which is always symmetric, is provided by

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\,\partial^{\mu}\phi(x)}\,\partial^{\nu}\phi(x) \,-\, \mathcal{L}(x)\,g^{\mu\nu} \;=\; T^{\nu\mu}$$

in such a manner that we obtain the classical hamiltonian

$$H = \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\,\left[\,\Pi^2(t,\mathbf{x}) + |\boldsymbol{\nabla}\phi(t,\mathbf{x})|^2 - \mu^2\,\phi^2(t,\mathbf{x}) + \tfrac{1}{2}\,\lambda\,\phi^4(t,\mathbf{x})\,\right]$$

which represents the total energy of the system, as well as the total momentum

$$\mathbf{P} = \int \mathrm{d}\mathbf{x}\;\Pi(t,\mathbf{x})\cdot\boldsymbol{\nabla}\phi(t,\mathbf{x})$$

Notice that the mass term in the Hamiltonian just corresponds to an imaginary tachyon-like mass $i\mu$

3. The classical field configurations which minimize the total energy are required to satisfy

$$\Pi(t,\mathbf{x}) = \dot{\phi}(t,\mathbf{x}) = 0 \qquad\qquad \boldsymbol{\nabla}\phi(t,\mathbf{x}) = 0$$

which yields

$$\phi = \text{constant}$$

and thereby

$$\delta H/\delta\phi = \,-\,\mu^2\,\phi + \lambda\,\phi^3 = 0 \;\Rightarrow\; \phi = 0 \,\vee\, \phi^2 = \frac{\mu^2}{\lambda} \,\equiv\, v^2 > 0$$

in such a manner that

$$H(\phi = 0) = 0 \qquad\qquad H(v) = \,-\,\tfrac{1}{4}\,\mu^2\,v^2\int \mathrm{d}\mathbf{x} < 0$$

that means that the two constant field configurations that minimize the energy density are

$$\phi_{\pm} = \,\pm\,v$$

which are evidently $\mathbb{Z}_2$ symmetric

4. After setting

$$\phi(x) = v + \varphi(x) \qquad\qquad \dot{\varphi}(x) = \Pi(x)$$

we readily obtain

$$\begin{aligned}
\mathcal{L}_\varphi &= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{2}\,\mu^2\,[\,\pm\,v + \varphi\,(x)\,]^2 - \tfrac{1}{4}\,\lambda\,[\,\pm\,v + \varphi\,(x)\,]^4 \\
&= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{2}\,\mu^2\,v^2 + \tfrac{1}{2}\,\mu^2\,\varphi^2\,(x) \pm \mu^2\,v\,\varphi\,(x) \\
&- \tfrac{1}{4}\,\lambda\Big(v^4 \pm 4v^3\varphi\,(x) + 6v^2\varphi^2\,(x) \pm 4v\,\varphi^3\,(x) + \varphi^4\,(x)\Big) \\
&= \tfrac{1}{2}\,\partial_\nu\,\varphi\,(x)\,\partial^\nu\,\varphi\,(x) + \tfrac{1}{4}\,\lambda\,v^4 - \mu^2\,\varphi^2\,(x) \\
&\mp\ \lambda\,v\,\varphi^3\,(x) - \tfrac{1}{4}\,\lambda\,\varphi^4\,(x)
\end{aligned}$$

and thereby

$$H = H_0 + H_{\rm int}$$

$$H_0 = \tfrac{1}{2}\int \mathrm{d}\mathbf{x}\ \big[\,\Pi^2(t,\mathbf{x}) + |\boldsymbol{\nabla}\varphi(t,\mathbf{x})|^2 + 2\mu^2\,\varphi^2(t,\mathbf{x}) - \mu^4/2\lambda\,\big]$$

$$H_{\rm int} = \int \mathrm{d}\mathbf{x}\ \Big[\,\tfrac{1}{4}\,\lambda\,\varphi^4(t,\mathbf{x}) \pm\ \mu\,\sqrt{\lambda}\,\varphi^3\,(t,\mathbf{x})\,\Big]$$

It follows that the shifted field $\varphi\,(t,\mathbf{x})$ has a mass $\mu\sqrt{2}$, a negative and divergent zero point energy does appear at the classical level, viz.

$$E_0 \;=\; -\,\frac{\mu^4}{4\lambda}\int \mathrm{d}\mathbf{x}$$

while the interacting Hamiltonian is unstable as it is not bounded from below, owing to the cubic term in the shifted field. Finally, the $\mathbb{Z}_2$ symmetry is broken.

0.9 Thursday 10 September 2009

TEORIA DEI CAMPI

1. Making use of the rules of correspondence we construct the matrix element

$$
\begin{aligned}
i\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}') &= -\,\mathrm{i}e^2\,\varepsilon_\mu^{A'}(k')\,\bar{u}_{r'}(\mathbf{p}')\,M^{\mu\nu}\,\varepsilon_\nu^{A}(k)\,u_r \\
&= \bar{u}_{r'}(\mathbf{p}')\,(-ie\,\gamma^\mu)\,\varepsilon_\mu^{A'}(k')\,S\,(p+k)\,(-ie\,\gamma^\nu)\,\varepsilon_\nu^{A}(k)\,u_r \\
&+ [\text{photon exchange} \quad k \leftrightarrow -k' \qquad A \leftrightarrow A' \qquad \mu \leftrightarrow \nu\,] \\
M^{\mu\nu} &= \frac{\gamma^\mu\,(\not p + \not k + m_e)\,\gamma^\nu}{(p+k)^2 - m_e^2} + \frac{\gamma^\nu\,(\not p - \not k' + m_e)\,\gamma^\mu}{(p-k')^2 - m_e^2}
\end{aligned}
$$

where the initial electron four momentum is $p^\lambda = (m_e, 0, 0, 0)$. Since $p^2 = m_e^2$ and $k^2 = 0$ we can simplify the above denominators as

$$
(p+k)^2 - m_e^2 = 2\,p^\rho\,k_\rho = 2\,k\,m_e
$$

$$
(p-k')^2 - m_e^2 = -\,2\,p^\rho\,k'_\rho = -\,2\,k'\,m_e
$$

while the numerators can be rearranged using a little Dirac algebra

$$
\begin{aligned}
(\not p + m_e)\,\gamma^\nu\,u_s(p) &= (2p^\nu - \gamma^\nu\,\not p + \gamma^\nu\,m_e)\,u_s(p) \\
&= 2p^\nu\,u_s(p) - \gamma^\nu\,(\not p - m_e)\,u_s(p) \\
&= 2p^\nu\,u_s(p)
\end{aligned}
$$

By doing the same for both numerators we get

$$
M^{\mu\nu} = \frac{\gamma^\mu\,\not k\,\gamma^\nu + 2\gamma^\mu\,p^\nu}{2\,k\,m_e} + \frac{\gamma^\nu\,\not k'\,\gamma^\mu - 2\,\gamma^\nu\,p^\mu}{2\,k'\,m_e}
$$

in such a manner that by putting altogether we can write

$$
i\,\mathcal{M}_{rr'}^{AA'}(\mathbf{k},\mathbf{k}') = \frac{-\,i\,e^2}{2\,k\,k'\,m_e}\,\bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\,M^{AA'}\,u_r
$$

where

$$
\begin{aligned}
M^{AA'} &= k'\,\varepsilon_\mu^{A'}(k')\,[\,\gamma^\mu\,\not k\,\gamma^\nu + 2\gamma^\mu\,p^\nu\,]\,\varepsilon_\nu^{A}(k) \\
&+ k\,\varepsilon_\nu^{A}(k)\,[\,\gamma^\nu\,\not k'\,\gamma^\mu - 2\,\gamma^\nu\,p^\mu\,]\,\varepsilon_\mu^{A'}(k') \\
&= k'\,\not\varepsilon^{A'}(k')\,\not k\,\not\varepsilon^{A}(k) + k\,\not\varepsilon^{A}(k)\,\not k'\,\not\varepsilon^{A'}(k')
\end{aligned}
$$

in which we have taken properly into account that

$$
\mathbf{p} = 0 \qquad \varepsilon_0^{A}(k) = 0 \qquad \varepsilon_0^{A'}(k') = 0 \qquad (A, A' = 1, 2)
$$

2. In calculating the probability of the process we have to average over the initial electron spin index r and sum over the final electron spin index r', so that we obtain

$$
\begin{aligned}
\tfrac{1}{2}\sum_{r,r'} \left| \mathcal{M}^{AA'}_{rr'}(\mathbf{k},\mathbf{k}') \right|^2 &= \frac{e^4}{8\,(k\,k'\,m_e)^2} \\
\times \quad \sum_{r,r'} \Big(\bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\, M^{AA'}\, u_r \Big)^* \, \bar{u}_{r'}(\mathbf{k}-\mathbf{k}')\, M^{AA'}\, u_r & \\
= \quad \frac{e^4}{8\,(k\,k'\,m_e)^2} \sum_{r,r'} \bar{u}_r\, \bar{M}^{AA'}\, u_{r'}(\mathbf{p}')\, \bar{u}_{r'}(\mathbf{p}')\, M^{AA'}\, u_r & \\
= \quad \frac{e^4}{8\,(k\,k'\,m_e)^2} \sum_{r} u_r\, \bar{u}_r\, \bar{M}^{AA'} \sum_{r'} u_{r'}(\mathbf{p}')\, \bar{u}_{r'}(\mathbf{p}')\, M^{AA'} & \\
= \quad \frac{e^4}{8\,(k\,k'\,m_e)^2}\, \mathrm{tr}\left[(\not{p} + m_e)\, \bar{M}^{AA'}\, (\not{p}' + m_e)\, M^{AA'} \right] & \\
= \quad \frac{e^4}{8\,(k\,k'\,m_e)^2}\, \mathrm{tr}\left[(\not{p}' + m_e)\, M^{AA'}\, (\not{p} + m_e)\, \bar{M}^{AA'} \right] & \\
= \quad \frac{e^4}{8\,(k\,k'\,m_e)^2}\, \mathrm{tr}\; Q^{AA'} &
\end{aligned}
$$

where

$$
\begin{aligned}
M^{AA'} &= k'\,\not{\epsilon}^{\,A'}(k')\,\not{k}\,\not{\epsilon}^{\,A}(k) + k\,\not{\epsilon}^{\,A}(k)\,\not{k}'\,\not{\epsilon}^{\,A'}(k') \\
\bar{M}^{AA'} &= k'\,\not{\epsilon}^{\,A}(k)\,\not{k}\,\not{\epsilon}^{\,A'}(k') + k\,\not{\epsilon}^{\,A'}(k')\,\not{k}'\,\not{\epsilon}^{\,A}(k) \\
Q^{AA'} &= (\not{p}' + m_e)\, M^{AA'}\, (\not{p} + m_e)\, \bar{M}^{AA'}
\end{aligned}
$$

0.10 Thursday 9 July 2009

TEORIA DEI CAMPI **1.**

1. The canonical energy momentum tensor of the scalar field, which is always symmetric, is provided by

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\,\partial^{\mu}\phi(x)}\,\partial^{\nu}\phi(x) \;-\; \mathcal{L}(x)\,g^{\mu\nu} \;=\; T^{\nu\mu}$$

$$T^{\mu\nu} = \partial^{\mu}\phi(x)\,\partial^{\nu}\phi(x)$$

$$-\tfrac{1}{2}\,g^{\mu\nu}\left[\,\partial_{\rho}\phi(x)\,\partial^{\rho}\phi(x) - m^2\,\phi^2(x) - \frac{\lambda}{12}\,\phi^4(x)\right]$$

and fulfills

$$g_{\mu\nu}\,T^{\mu\nu} = T^{\mu}_{\ \ \mu} = \,-\,\partial_{\rho}\phi(x)\,\partial^{\rho}\phi(x) + 2\,m^2\,\phi^2(x) + \frac{\lambda}{6}\,\phi^4(x)$$

2. The infinitesimal dilatation transformations read

$$\delta x^{\mu} = x^{\mu}\,\delta\alpha \qquad\qquad \Delta\phi(x) = \,-\,\phi(x)\,\delta\alpha$$

so that we find, in accordance with Noether theorem,

$$\begin{aligned} J^{\mu}(x) &\equiv T^{\mu}_{\ \ \nu}(x)\,x^{\nu} + \frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\phi(x)}\,\phi(x) \\ &= \left[\frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\phi(x)}\,\partial_{\nu}\phi(x) \;-\; \mathcal{L}(x)\,\delta^{\mu}_{\nu}\right]x^{\nu} + \phi(x)\,\partial^{\mu}\phi(x) \end{aligned}$$

which satisfy

$$\begin{aligned} \partial_{\mu}\,J^{\mu}(x) &= T^{\mu}_{\ \ \mu}(x) + \tfrac{1}{2}\,\Box\,\phi^2(x) \\ &= 2\,\partial^{\mu}\phi(x)\,\partial_{\mu}\phi(x) - 4\,\mathcal{L}(x) + \phi(x)\,\Box\,\phi(x) \\ &= 2\,m^2\,\phi^2(x) + \frac{\lambda}{3!}\,\phi^3(x) - m^2\,\phi^2(x) - \frac{\lambda}{3!}\,\phi^3(x) \\ &= m^2\,\phi^2(x) \;\neq\; 0 \end{aligned}$$

We see that the failure for the continuity equation of the dilatation current is actually due to the only dimensionful parameter present in the lagrangian, *i.e.* the mass of the real scalar field.

3. If we take

$$\Theta^{\mu}_{\ \ \nu}(x) = T^{\mu}_{\ \ \nu}(x) + a\,(\,\partial^{\mu}\,\partial_{\nu} - \delta^{\mu}_{\nu}\,\Box\,)\;\phi^2(x)$$

in such a manner that the continuity equation keeps to hold true, *viz.,*

$$\partial_\mu \Theta^{\mu}{}_{\nu}(x) \;\equiv\; \partial_\mu T^{\mu}{}_{\nu}(x) \;=\; 0$$

then in the massless case $m = 0$ we find

$$\begin{aligned} \Theta^{\mu}{}_{\mu}(x) &= -\,\partial_\rho \phi(x)\,\partial^\rho \phi(x) + \frac{\lambda}{6}\,\phi^4(x) \\ &- \; 6a\left[\partial_\lambda \phi(x)\,\partial^\lambda \phi(x) - \frac{\lambda}{6}\,\phi^4(x)\right] \end{aligned}$$

which yields, in the massless case,

$$\Theta^{\mu}{}_{\mu} = 0 \quad \Leftrightarrow \quad a = \,-\,\frac{1}{6}$$

Hence one is led to the suitable definition of the *new improved* energy-momentum tensor for the scalar field : namely,

$$\Theta^{\mu}{}_{\nu}(x) = T^{\mu}{}_{\nu}(x) - \tfrac{1}{6}\,(\,\partial^\mu\,\partial_\nu - \delta^\mu_\nu\,\Box\,)\;\phi^2(x)$$

4. If we now define the new dilatation current

$$j^{\mu}(x) = x^{\nu}\;\Theta^{\mu}{}_{\nu}(x)$$

we see that

$$\partial_\mu\, j^{\mu}(x) = g_{\lambda\nu}\,\Theta^{\lambda\nu}(x) = T^{\mu}{}_{\mu}(x) + \tfrac{1}{2}\,\Box\,\phi^2(x) = \partial_\mu\, J^{\mu}(x)$$

and consequently, in the massless case $m = 0$, the divergenceless of the new improved dilatation current $j^{\mu}(x)$ is equivalent to the tracelessness condition for the new improved energy-momentum tensor $\Theta^{\lambda\nu}(x)$. The new improved dilatation current is related to the previous canonical Noether current by

$$\begin{aligned} j^{\mu}(x) &= T^{\mu}{}_{\nu}(x)\,x^{\nu} - \tfrac{1}{6}\,x^{\nu}\,(\,\partial^\mu\,\partial_\nu - \delta^\mu_\nu\,\Box\,)\;\phi^2(x) \\ &= J^{\mu}(x) - \tfrac{1}{2}\,\partial^\mu\,\phi^2(x) - \tfrac{1}{6}\,x^{\nu}\,(\,\partial^\mu\,\partial_\nu - \delta^\mu_\nu\,\Box\,)\;\phi^2(x) \\ &= J^{\mu}(x) + \tfrac{1}{6}\,\partial_\nu\,(\,x^{\mu}\,\partial^{\nu} - x^{\nu}\,\partial^{\mu}\,)\,\phi^2(x) \end{aligned}$$

0.11 Thursday 9 July 2009

TEORIA DEI CAMPI

Consider the electron-proton scattering $e^- p \;\rightarrow\; p e^-$ and suppose the proton to be structureless. Then the Feynman rules give at once the lowest order $O(e^2)$ amplitude : namely,

$$\mathrm{i}\,\mathcal{M} \;=\; \bar{u}_{r'}(p_1')\,\gamma^\mu\,u_r(p_1)\,\frac{\mathrm{i}\,e^2\,g_{\mu\nu}}{(p_1-p_1')^2}\,\bar{u}_{s'}(p_2')\,\gamma^\nu\,u_s(p_2)$$

Taking the square modulus as well as the average over the incoming particle spin and the sum over the final particle spin we find

$$\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;=\; \left(\frac{e^2}{2t}\right)^2 \quad g_{\mu\nu}\,g_{\rho\sigma}\;\times$$
$$\mathrm{tr}\;[\,(\not p_1' + m_e)\,\gamma^\mu\,(\not p_1 + m_e)\,\gamma^\rho\,]\;\mathrm{tr}\;[\,(\not p_2' + m_\mu)\,\gamma^\nu\,(\not p_2 + m_\mu)\,\gamma^\sigma\,]$$

in which the momentum trasfer Mandelstam's variable $t = (\,p_1' - p_1\,)^2$ has been employed. Hence, calculating traces by means of the formulæ

$$\begin{aligned} \mathrm{tr}\;(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} \;=\; 4\,g^{\mu\nu} \\ \mathrm{tr}\;\left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right) &= 4\left(g^{\kappa\lambda}\,g^{\mu\nu} \;-\; g^{\kappa\mu}\,g^{\lambda\nu} \;+\; g^{\kappa\nu}\,g^{\lambda\mu}\right) \end{aligned} \tag{1}$$

the trace of an odd number of gamma matrices being null, we readily come to the exact expression

$$\sum_{r,r'}\sum_{s,s'}\tfrac{1}{4}\,|\,\mathcal{M}\,|^2 \;=\; \frac{8\,e^4}{t^2}\;\times$$
$$\Big[\,(p_1\cdot p_2)\,(p_1'\cdot p_2') + (p_1\cdot p_2')\,(p_1'\cdot p_2) - (p_1\cdot p_1')\,m_p^2$$
$$-\;(p_2\cdot p_2')\,m_e^2 + 2\,m_p^2\,m_e^2\,\Big]$$

Moreover, taking into account that $m_e \ll m_p$, the above quantity can be approximated as

$$\tfrac{1}{4}\sum_{r,r'}\sum_{s,s'}|\,\mathcal{M}\,|^2 \;\simeq\; \frac{8\,e^4}{t^2}\;\times$$
$$[\,(p_1\cdot p_2)\,(p_1'\cdot p_2') + (p_1\cdot p_2')\,(p_1'\cdot p_2) - (p_1\cdot p_1')\,m_p^2\,]$$

Consider now the collision process in the initial proton rest frame, keeping the electron mass $m_e \ll m_p$ and treating the proton mass as very large. Then, if we disregard the proton recoil, the kinematics reads

$$\left.\begin{array}{lll} \text{incoming electron}: & \mathbf{p}_1 = \mathbf{p} & E_1 = \sqrt{\mathbf{p}^2 + m_e^2} = E \\ \text{incoming proton}: & \mathbf{p}_2 = 0 & E_2 = m_p \\ \text{outgoing proton}: & \mathbf{p}_2' \approx 0 & E_2' \approx m_p \\ \text{outgoing electron}: & \mathbf{p}_1' = \mathbf{p}' & E_1' \approx E \ll m_p \end{array}\right\}$$

with $|\,\mathbf{p}\,| = p \approx |\,\mathbf{p}'\,|$, whence

$$\begin{array}{l} p_1 \cdot p_2 \ = \ E\,m_p \ \approx \ p_1' \cdot p_2' \ \approx \ p_1 \cdot p_2' \ \approx \ p_1' \cdot p_2 \\ p_1 \cdot p_1' = E^2 - p^2 \cos\theta \qquad t = (p_1 - p_1')^2 \approx -\,2p^2\,(1 - \cos\theta\,) \end{array}$$

$$\sum_{r,r'} \sum_{s,s'} \tfrac{1}{4}\,|\,\mathcal{M}\,|^2 \ \approx \ \frac{e^4\,m_p^2}{2p^4\,\sin^4(\theta/2)}\,\left[\,2\,E^2 - (\,E^2 - p^2\cos\theta - m_e^2\,)\,\right]$$

If one of the two incident particles is sufficiently heavy, like the proton in the present example, so that its state does not change after the collision, then its role in the process is equivalent to a fixed target for which recoil can be disregarded. Turning back to the main basic formula for the differential cross section, in the present case of a 2-particle final state with one very heavy particle of mass M we can use the kinematics where θ is now the scattering angle of the light particle with mass m in the heavy particle rest frame : namely,

$$\begin{aligned} & I \equiv \sqrt{(p_1 \cdot p_2)^2 - m^2 M^2} = |\,\mathbf{p}\,|\,M \\ & \prod_{k=1}^{2} \int \frac{\mathrm{d}\,\mathbf{p}_k'}{(2\pi)^3\,2\omega(\,\mathbf{p}_k'\,)}\ (2\pi)^4\,\delta\,(p_1' + p_2' - p_1 - p_2) \\ & = \int \mathrm{d}\,\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}p\,p^2}{16\pi^2 M E\,(p)}\ \delta\left(E_{p'}' - E_p\right) \\ & = \int \mathrm{d}\,\Omega\,(\phi,\theta) \int_0^\infty \frac{\mathrm{d}E\,p\,(\,E\,)}{16\pi^2 M}\ \delta\,(E' - E) \\ & = \frac{|\,\mathbf{p}'\,|}{16\,\pi^2\,M} \int \mathrm{d}\,\Omega\,(\phi,\theta) \ \approx \ \frac{|\,\mathbf{p}\,|}{16\,\pi^2\,M} \int \mathrm{d}\,\Omega\,(\phi,\theta) \end{aligned}$$

Thus, according to the main formula for the differential cross section and the above fixed target (FT) kinematics, as well as the related final 2-particle phase space integration, we eventually come to the remarkable and simple expression

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{FT}} = \frac{|\,\mathcal{M}\,(s,t,u)\,|^2}{64\,\pi^2\,M^2}$$

Inserting the above obtained initial spin averaged and final spin summed amplitude and setting $M = m_p$ yields

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^2}{4\,|\,\mathbf{p}\,|^2\,\beta^2\,\sin^4(\theta/2)}\,\left(1-\beta^2\,\sin^2\frac{\theta}{2}\right)$$

where $\beta \equiv |\,\mathbf{p}\,|\,/\,E$, which is the celebrated *Mott formula* for the Coulomb scattering of relativistic electrons. In the non-relativistic limit and for a fixed target of atomic number Z we readily recover the *Rutherford formula*. Actually, for $\beta = v\,/\,c$, $\mathbf{p} \approx m_e\,\mathbf{v}$, $E \approx m_e\,c^2$, we get the leading term

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} \overset{\beta\to 0}{\sim} \frac{Z^2\,\alpha^2\,(\hbar c)^2}{4\,m_e^2\,v^4\,\sin^4(\theta/2)}$$

0.12 Friday 12 June 2009

1. The chiral phase transformations are

$$\psi'(x) = [\cos\alpha - \mathrm{i}\,\sin\alpha\,\gamma_5\,]\,\psi(x)$$

$$\bar\psi'(x) = \bar\psi(x)\,[\cos\alpha - \mathrm{i}\,\sin\alpha\,\gamma_5\,]$$

which keep invariant the kinetic term. Hence we find

$$\begin{aligned} -M\,\bar\psi(x)\,\psi(x) - i\,M'\,\bar\psi(x)\,\gamma_5\,\psi(x) \;&= \\ -\,M\,\cos 2\alpha\,\bar\psi(x)\,\psi(x) -\, iM\,\sin 2\alpha\,\bar\psi(x)\,\gamma_5\,\psi(x) \\ -\,iM'\,\cos 2\alpha\,\bar\psi(x)\,\gamma_5\,\psi(x) +\, M'\,\sin 2\alpha\,\bar\psi(x)\,\psi(x) \end{aligned}$$

and thereby the pseudoscalar mass term is removed iff

$$\tan 2\alpha \;=\; -\,\frac{M'}{M} = \frac{2t}{1-t^2}$$

$$t = \tan\alpha = \frac{M - \sqrt{M(M'+M)}}{M'} \qquad\qquad (\,-1 < t < 0\,)$$

in such a manner that the new mass term becomes

$$\widetilde{M} = M\sec 2\alpha = M\,\frac{1+t^2}{1-t^2}$$

Now, if we set

$$M_\pm \;\equiv\; M\,\pm\,M'$$

then we can write

$$\widetilde{M} \;=\; \frac{M^2\left(2\sqrt{MM_+}\,-M_-\right) - MM_+^2}{M_+\,M_- - M\left(2\sqrt{MM_+}\,-M_+\right)} \;>\;0$$

2. If we trust in general relativity and in quantum field theory up to the Planck scale but not beyond, it turns out to be quite natural to cut−off the loop integrations at a very high wave number of the order $K \simeq (8\pi G_N)^{-1/2}$. Consider therefore spinor tadpole in the ultra−violet momentum cut−off regularisation

$$\begin{aligned} &\mathtt{reg}\,\Sigma_1(\not{0}) \;= \\ &= \;\frac{-ig}{(2\pi)^4}\int \mathrm{d}\,\mathbf{p}\;\theta(\,K^2-\mathbf{p}^2\,)\int_{-\infty}^{\infty}\frac{\mathrm{tr}\,(\,\not{p}+M\,)\,\mathrm{d}p_0}{p_0^2-\mathbf{p}^{\,2}-M^{\,2}+i\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{4igM}{(2\pi)^4}\int \mathrm{d}\mathbf{p}\;\theta(K^2-\mathbf{p}^2)\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{p_0^2-\mathbf{p}^2-M^2+i\varepsilon} \\
&= -\frac{gM}{\pi^2}\int_0^K \mathrm{d}p\;p^2\;\left(p^2+M^2\right)^{-1/2} \\
&= -\frac{gM}{2\pi^2}\,K^2\left\{\sqrt{1+M^2/K^2}-M^2/K^2\right. \\
&\times\;\left.\left[\ln\,K/M+\ln\left(1+\sqrt{1+M^2/K^2}\right)\right]\right\} \\
&= \frac{gM}{2\pi^2}\left\{-\,K^2+M^2\left[\ln\frac{K}{M}-\frac{1}{2}+\ln 2+O\left(\frac{M}{K}\right)^2\right]\right\}
\end{aligned}
$$

which means, as expected, that we have again a quadratic divergence and a logarithmic divergence at the Planck scale.

Dimensional regularisation yields

$$
\texttt{reg}\,\Sigma_1(\emptyset) \;=\; -\,ig\int_p\frac{\mathrm{tr}\,[\,(\not p+M\,)\,]}{p^2-M^2+i\varepsilon} \;=\; \int_p\frac{-\,4igM}{p^2-M^2+i\varepsilon}
$$

where we have set for the sake of brevity

$$
\int_p \stackrel{\mathrm{def}}{=} \mu^{2\epsilon}\int\frac{d^{2\omega}p}{(2\pi)^{2\omega}} \qquad [\,\epsilon=2-\omega\,]
$$

Now we can perform the Wick rotation with $p_0=ip_4$ and get

$$
\texttt{reg}\,\Sigma_1(\emptyset)=\int_p\frac{-\,4gM}{p_E^2+M^2}
$$

Turning to spherical polar coordinates in 2ω dimensions we find

$$
\begin{aligned}
\texttt{reg}\,\Sigma_1(\emptyset) &= -\,\frac{8gM\,\pi^{\omega}\,\mu^{4-2\omega}}{\Gamma(\omega)\,(2\pi)^{2\omega}}\int_0^{\infty}\frac{p_E\,(p_E^2)^{\omega-1}}{p_E^2+M^2}\;\mathrm{d}p_E \\
&= -\,\frac{4gM\,\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\int_0^{\infty}\frac{\mathrm{d}q\;q^{\omega-1}}{q+M^2} \\
&= -\,\frac{4gM\,\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\int_0^{\infty}\mathrm{d}q\;q^{\omega-1}\int_0^{\infty}\mathrm{d}t\;\exp\{-\,tq-tM^2\} \\
&= -\,\frac{4gM\,\mu^{4-2\omega}}{(4\pi)^{\omega}\,\Gamma(\omega)}\int_0^{\infty}\mathrm{d}t\;\mathrm{e}^{-\,tM^2}\int_0^{\infty}\mathrm{d}q\;q^{\omega-1}\,\mathrm{e}^{-tq} \\
&= -\,\frac{4gM\,\mu^{4-2\omega}}{(4\pi)^{\omega}}\int_0^{\infty}\mathrm{d}t\;t^{-\omega}\,\mathrm{e}^{-\,tM^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\,\frac{gM^3}{4\pi^2}\,\Gamma(1-\omega)\,(4\pi\mu^2/M^2)^{2-\omega} \\
&= \frac{gM^3}{4\pi^2}\left\{\frac{1}{\epsilon}+\psi(2)+O\,(\epsilon)\right\}\left(1+\epsilon\,\ln\frac{4\pi\mu^2}{M^2}\;+\;\cdots\right) \\
&= \frac{gM^3}{4\pi^2\,\epsilon}+\frac{gM^3}{4\pi^2}\left[\ln\frac{4\pi\mu^2}{M^2}+\psi(2)\right]\;+\;\text{irrelevant}
\end{aligned}
$$

which shows that the large wave number quadratic divergence

$$
-\,gMK^2/2\pi^2
$$

does not appear at all in the dimensional regularisation, while the logarithmic divergences coincide in the two regularisation schemes provided

$$
\ln\frac{K^2}{4\pi\mu^2}=1+\psi(2)-\ln 4
$$

that is

$$
K^2=\pi\mu^2\,\mathrm{e}^{2-\mathbf{C}}
$$

0.13 Friday 29 May 2009

TEORIA DEI CAMPI **1.**

1. In natural units $\hbar = c = 1$ we have $[\,A_\mu\,] = [\,\psi\,] =$ eV, while in C.G.S. e.m. Gauss physical units $[\,A_\mu\,] = [\,\psi\,] =$ cm^{-1}

2. The Euler-Lagrange field equations read

$$\tfrac{1}{2}\,\varepsilon^{\mu\rho\sigma}\,F_{\rho\sigma}(x) \equiv \widetilde{F}^{\mu}(x) = -\,\frac{q}{\varkappa}\,\bar{\psi}(x)\,\gamma^{\mu}\,\psi(x)$$

$$(-\mathrm{i}\partial\!\!\!/ + M)\,\psi(x) = q\,A\!\!\!/(x)\,\psi(x)$$

or equivalently with $\widetilde{F}^{\mu} = (B, \mathbf{E})$

$$\begin{aligned} B(t,\mathbf{r}) &= (\,-\,q/\varkappa\,)\,\psi^{\dagger}(x)\,\psi(x) \\ E_x(t,\mathbf{r}) &= (\,-\,q/\varkappa\,)\,\psi^{\dagger}(x)\,\sigma_3\,\psi(x) \\ E_y(t,\mathbf{r}) &= (\,q/\varkappa\,)\,\psi^{\dagger}(x)\,\sigma_2\,\psi(x) \end{aligned}$$

which show that the field strength does not propagate within the planar Chern-Simons spinor electrodynamics.

3. The gauge transformations are U(1) local phase transformations, *i.e.* ,

$$\psi(x) \;\to\; \psi'(x) = \psi(x)\,\mathrm{e}^{\mathrm{i}q\alpha(x)} \qquad\qquad \bar{\psi}(x) \;\to\; \bar{\psi}'(x) = \bar{\psi}(x)\,\mathrm{e}^{-\,\mathrm{i}q\alpha(x)}$$

$$A_\mu(x) \;\to\; A'_\mu(x) = A_\mu(x) + \partial_\mu\,\alpha(x)$$

In the infinitesimal form they read

$$\delta\psi(x) = \mathrm{i}q\alpha(x) = -\,\delta\bar{\psi}(x) \qquad\qquad \delta\,A_\mu(x) = \partial_\mu\,\delta\,\alpha(x)$$

so that we find

$$\begin{aligned} \delta\mathcal{L} &= \tfrac{1}{4}\,\varepsilon^{\mu\rho\sigma}\,F_{\rho\sigma}(x)\,\partial_\mu\,\delta\,\alpha(x) \\ &= \tfrac{1}{4}\,\varepsilon^{\mu\rho\sigma}\,\partial_\mu\Big(F_{\rho\sigma}(x)\,\delta\,\alpha(x)\Big) \end{aligned}$$

the very last term being a boundary term. By making use of the Gauß theorem we can recast the integral of the boundary term as

$$\int_{\mathrm{C}} \mathrm{d}\mathbf{r}\;\tfrac{1}{2}\;[\,B(t_f,\mathbf{r})\,\delta\,\alpha(t_f,\mathbf{r}) - B(t_i,\mathbf{r})\,\delta\,\alpha(t_i,\mathbf{r})\,]$$

$$+\;\; \tfrac{1}{2}\lim_{R\to\infty}\,R\int_{t_i}^{t_f}\mathrm{d}t\int_0^{2\pi}\mathrm{d}\phi\;\widehat{\mathbf{n}}\cdot\widetilde{\mathbf{F}}(t,R,\phi)\,\delta\,\alpha(t,R,\phi)$$

where R is the ray of a very large circle C centered at the origin and $\widehat{\mathbf{n}}$ is the exterior unit vector normal to the circle. Hence, the gauge invariance of the Chern-Simons action is recovered iff

$$\delta\,\alpha(t_f,\mathbf{r}) = \delta\,\alpha(t_i,\mathbf{r}) = 0 \qquad\qquad \lim_{R\to\infty} R\,\widehat{\mathbf{n}}\cdot\widetilde{\mathbf{F}}(t,R,\phi)\,\delta\,\alpha(t,R,\phi) = 0$$

4. For a null electromagnetic coupling $q = 0$ and in the massless case $M = 0$ we get

$$\psi_{\mathbf{p}}(x) = \psi_{\mathbf{p}}(t,\mathbf{r}) = \Gamma(\,p_0,\mathbf{p}\,)\,\exp\{\,-\,i\,\omega_{\mathbf{p}}\,t + i\,\mathbf{p}\cdot\mathbf{r}\}$$

so that

$$\begin{pmatrix} -ip_y & \omega_{\mathbf{p}} - p_x \\ \omega_{\mathbf{p}} + p_x & ip_y \end{pmatrix} \Gamma(p_0,\mathbf{p}) = 0$$

which leads to $p_0 = \pm\,\omega_{\mathbf{p}} = \pm\,|\,\mathbf{p}\,|$. For the particle solutions it is convenient to set

$$u(\mathbf{p}) = (\,p - p_x\,)^{-1/2} \begin{pmatrix} p - p_x \\ i\,p_y \end{pmatrix} \qquad\qquad (\,p \equiv \omega_{\mathbf{p}} = |\,\mathbf{p}\,|\,)$$

$$u_{\mathbf{p}}(t,\mathbf{r}) = [\,(2\pi)^2\,2p\,]^{-1/2}\,u(\mathbf{p})\,\exp\{\,-\,i\,p\,t + i\,\mathbf{p}\cdot\mathbf{r}\}$$

which satisfy

$$u^{\dagger}(\mathbf{p})\,u(\mathbf{p}) = 2p \qquad\qquad \bar{u}(\mathbf{p})\,u(\mathbf{p}) = 0$$

$$\int \mathrm{d}\mathbf{r}\;u^{\dagger}_{\mathbf{q}}(t,\mathbf{r})\,u_{\mathbf{p}}(t,\mathbf{r}) = \delta(\mathbf{p}-\mathbf{q})$$

$$\int \mathrm{d}\mathbf{p}\;u^{\dagger}_{\mathbf{p}}(t,\mathbf{r}')\,u_{\mathbf{p}}(t,\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$$

For the antiparticle solutions with $\omega_{\mathbf{p}} = -\,p$ we set

$$v(\mathbf{p}) = (\,p + p_x\,)^{-1/2} \begin{pmatrix} -\,i\,p_y \\ p + p_x \end{pmatrix} \qquad\qquad (\,p \equiv |\,\mathbf{p}\,|\,)$$

$$v_{\mathbf{p}}(t,\mathbf{r}) = [\,(2\pi)^2\,2p\,]^{-1/2}\,v(\mathbf{p})\,\exp\{\;i\,p\,t - i\,\mathbf{p}\cdot\mathbf{r}\}$$

which satisfy

$$v^{\dagger}(\mathbf{p})\,v(\mathbf{p}) = 2p \qquad\qquad \bar{v}(\mathbf{p})\,v(\mathbf{p}) = 0$$

$$\int \mathrm{d}\mathbf{r}\;v^{\dagger}_{\mathbf{q}}(t,\mathbf{r})\,v_{\mathbf{p}}(t,\mathbf{r}) = \delta(\mathbf{p}-\mathbf{q})$$

$$\int \mathrm{d}\mathbf{p}\; v^{\dagger}_{\mathbf{p}}(t,\mathbf{r}')\, v_{\mathbf{p}}(t,\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$$

together with

$$u^{\dagger}(\mathbf{p})\, v(-\mathbf{p}) = 0 = \bar{u}(\mathbf{p})\, v(\mathbf{p}) \qquad v^{\dagger}(\mathbf{p})\, u(-\mathbf{p}) = 0 = \bar{v}(\mathbf{p})\, u(\mathbf{p})$$
$$\int \mathrm{d}\mathbf{r}\; v^{\dagger}_{\mathbf{q}}(t,\mathbf{r})\, u_{\mathbf{p}}(t,\mathbf{r}) = 0 = \int \mathrm{d}\mathbf{r}\; u^{\dagger}_{\mathbf{q}}(t,\mathbf{r})\, v_{\mathbf{p}}(t,\mathbf{r})$$
$$\int \mathrm{d}\mathbf{p}\; v^{\dagger}_{\mathbf{p}}(t,\mathbf{r}')\, u_{\mathbf{p}}(t,\mathbf{r}) = 0 = \int \mathrm{d}\mathbf{p}\; u^{\dagger}_{\mathbf{p}}(t,\mathbf{r}')\, v_{\mathbf{p}}(t,\mathbf{r})$$

Hence, the complete orthonormal sets of the improper solutions of the Dirac equation can be obtained in the absence of electromagnetic fields

$$u_{\mathbf{p}}(t,\mathbf{r}) = [\,(2\pi)^2\, 2p\,(\,p-p_x\,)\,]^{-1/2} \begin{pmatrix} p-p_x \\ i\,p_y \end{pmatrix} \mathrm{e}^{i\,\mathbf{p}\cdot\mathbf{r}-i\,p\,t}$$
$$v_{-\mathbf{p}}(t,\mathbf{r}) = [\,(2\pi)^2\, 2p\,(\,p-p_x\,)\,]^{-1/2} \begin{pmatrix} i\,p_y \\ p-p_x \end{pmatrix} \mathrm{e}^{i\,\mathbf{p}\cdot\mathbf{r}+i\,p\,t}$$

Thus the quantized free Dirac field in a 2+1 dimensional Minkowski space-time reads

$$\Psi(t,\mathbf{r}) = \int \mathrm{d}\mathbf{p}\;\left[\,\mathrm{a}_{\mathbf{p}}\, u_{\mathbf{p}}(t,\mathbf{r}) \;+\; \mathrm{b}^{\dagger}_{\mathbf{p}}\, v_{\mathbf{p}}(t,\mathbf{r})\,\right]$$
$$\bar{\Psi}(t,\mathbf{r}) = \Psi^{\dagger}(t,\mathbf{r})\,\sigma_1 = \int \mathrm{d}\mathbf{p}\;\left[\,\mathrm{a}^{\dagger}_{\mathbf{p}}\, \bar{u}_{\mathbf{p}}(t,\mathbf{r}) \;+\; \mathrm{b}_{\mathbf{p}}\, \bar{v}_{\mathbf{p}}(t,\mathbf{r})\,\right]$$

where the creation and destruction Fermi operators do satisfy canonical anticommutation relations

$$\left\{\,\mathrm{a}_{\mathbf{p}}\,,\,\mathrm{a}^{\dagger}_{\mathbf{q}}\,\right\} = \delta(\mathbf{p}-\mathbf{q}) = \left\{\,\mathrm{b}_{\mathbf{p}}\,,\,\mathrm{b}^{\dagger}_{\mathbf{q}}\,\right\}$$

all the other anticommutators being equal to zero.

1. The kinetic differential operator for the gauge field is

$$\tfrac{1}{2}\,\varepsilon^{\lambda\mu\nu}\,\mathrm{i}\,k_\mu$$

which is not invertible, so that the gauge field propagator does not exist.

2. The most general gauge invariant and Poincaré invariant Action, of the utmost 2nd order in the gauge and spinor fields, is provided by the integral of the Lagrange density

$$\mathcal{L} = \mathcal{L}_{c\ell} \; + \; \Delta\,\mathcal{L}$$

where

$$\mathcal{L}_{c\ell} = \tfrac{1}{4}\,\varkappa\,\varepsilon^{\lambda\mu\nu}\,A_\lambda(x)\,F_{\mu\nu}(x) + \bar\psi(x)\,[\,\mathrm{i}\partial\!\!\!/ + q\,A\!\!\!/(x) - M\,]\,\psi(x)$$

$$\Delta\,\mathcal{L} = -\,\frac{C_1}{4M}\,F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{C_2}{2M}\,F_{\mu\nu}(x)\,\bar\psi(x)\,\sigma^{\mu\nu}\,\psi(x)$$

in which C_1 and C_2 are arbitrary constants while $\sigma^{\mu\nu}$ is the spin matrix for the spinor field, *i.e.* , $\sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\,[\,\gamma^\mu\,,\,\gamma^\nu\,]$.

3. The naïve expression for the planar photon self−energy, or planar QED vacuum polarization tensor, readily follows from the Feynman rules

$$\begin{aligned}
\mathrm{i}\,\Pi^{\mu\nu}(k) &= (-1)\,(\mathrm{i}\,q)^2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\;\mathrm{tr}\,\{\gamma^\mu\,S(p)\,\gamma^\nu\,S(p+k)\} \\
&= \frac{-\,q^2}{(2\pi)^3}\int \frac{\mathrm{tr}\,\{\gamma^\mu\,(p\!\!\!/ + M)\,\gamma^\nu\,(p\!\!\!/ + k\!\!\!/ + M)\}\;\mathrm{d}^3 p}{(p^2 - M^2 + \mathrm{i}\,\varepsilon)[\,(p+k)^2 - M^2 + \mathrm{i}\,\varepsilon\,]}
\end{aligned}$$

which exhibits a logarithmic divergence by naïve power counting.

4. We have to evaluate

$$\begin{aligned}
\Pi(\,k^2\,,\,M^2\,) &= \frac{-\,4\,q^2}{(4\pi)^{\frac{3}{2}}}\int_0^1 \frac{x(1-x)\,\Gamma(\frac{1}{2})\;\mathrm{d}x}{\sqrt{[\,M^2 - x(1-x)\,k^2\,]}} \\
&= \frac{q^2\,\partial}{\pi\,\partial\,k^2}\int_0^1 \mathrm{d}x\;\sqrt{M^2 - x(1-x)\,k^2}
\end{aligned}$$

After setting

$$R = a + bx + cx^2 \qquad\quad a = M^2 \qquad\quad c = k^2 = -\,b$$

we find

$$\Pi(k^2, M^2) = \frac{-\,q^2}{2\pi} \int_0^1 \frac{\mathrm{d}x}{\sqrt{R}}\, x(1-x)$$

and from I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Alan Jeffrey Editor, Academic Press, San Diego (CA) 1994, equations **2.264**2. and **2.264**3., p.100 we obtain

$$\Pi(k^2, M^2) \;=\; \frac{1}{8}\left(1+\frac{4M^2}{k^2}\right)\int_0^1 \frac{\mathrm{d}x}{\sqrt{R}} - \frac{5M}{4k^2}$$

and taking into account that from *ibidem*, equation **2.261**, p. 99 we have

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}x}{\sqrt{R}} &= \frac{1}{\sqrt{k^2}} \ln \frac{2M\sqrt{k^2}+k^2}{2M\sqrt{k^2}-k^2} && \left[\,k^2>0\,\right] \\ &= \frac{2}{\sqrt{-\,k^2}} \arcsin \frac{-\,k^2}{\sqrt{-\Delta}} && \left[\,k^2<0\,\right] \end{aligned}$$

where

$$\Delta = \frac{4M^{\,2}}{k^2} - 1$$

Thus we eventually get

$$\begin{aligned} \Pi(k^2, M^2) + \frac{5M}{4k^2} &= \frac{1}{4\sqrt{k^2}}\left(1+\frac{4M^2}{k^2}\right) \operatorname{Arcth} \frac{2M}{\sqrt{k^2}} && \left[\,k^2>0\,\right] \\ &= \frac{1}{4\sqrt{-\,k^2}}\left(1+\frac{4M^2}{k^2}\right) \arcsin \frac{-\,k^2}{\sqrt{-\Delta}} && \left[\,k^2<0\,\right] \end{aligned}$$

5. For $k^2 = 0$ we obtain

$$\Pi\,(\,0\,,\,M^{\,2}\,) = \frac{-\,q^2}{2\pi M}\int_0^1 x(1-x)\;\mathrm{d}x = \frac{-\,q^2}{12\pi M}$$

that gives rise to the Lagrangian, for $q = -\,e$,

$$\frac{-\,\alpha}{12M}\, F^{\mu\nu}(x)\, F_{\mu\nu}(x)$$

which is of the Maxwell form but for the overall coefficient $C_1 = \frac{1}{3}\,\alpha\,$, and for the mass parameter M, which is necessary in a *2+1* dimensional Minkowski space-time.

0.14 Thursday 22 January 2009

1. We have the following results :

1. the (passive) special Lorentz transformation, or even boost, in the Oz direction with velocity $v > 0$ towards the positive Oz axis

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$

with

$$\cosh\eta = (1-\beta^2)^{-1/2} \qquad \sinh\eta = \beta\,(1-\beta^2)^{-1/2} \qquad \beta = \frac{v_z}{c}$$

this 4×4 matrix belongs to the $\boldsymbol{\tau}_{\frac12\,\frac12}$ irreducible vector representation of the Lorentz group

2. the left-handed matrix belonging to the irreducible Weyl representation $\boldsymbol{\tau}_{\frac12\,0}$ is provided by

$$\Lambda_L(\eta) \equiv \exp\left\{\tfrac12\,\sigma_3\,\eta\right\} = \cosh(\eta/2) + \sigma_3\,\sinh(\eta/2)$$

3. The passive boost along the $Oz-$axis and belonging to the reducible representation $\boldsymbol{\tau}_{\frac12\,0} \oplus \boldsymbol{\tau}_{0\,\frac12}$ acting upon Dirac spinors is realized by the 4×4 matrix

$$\Lambda_{\frac12}(\eta) = \exp\left\{-\,\mathrm{i}\,S_{03}\,\omega^{03}\right\} = \cosh\frac{\eta}{2} - \gamma^0\gamma^3\sinh\frac{\eta}{2}$$

in which

$$\mathrm{i}\,S_{\mu\nu} \;=\; \tfrac14\,[\,\gamma^\rho\,,\,\gamma^\sigma\,]\;g_{\mu\rho}\,g_{\nu\sigma}$$

so that, using the chiral or spinor Weyl representation for the gamma matrices we come to

$$\Lambda_{\frac12}(\eta) = \begin{pmatrix} \mathrm{ch}\frac{\eta}{2} + \mathrm{sh}\frac{\eta}{2} & 0 & 0 & 0 \\ 0 & \mathrm{ch}\frac{\eta}{2} - \mathrm{sh}\frac{\eta}{2} & 0 & 0 \\ 0 & 0 & \mathrm{ch}\frac{\eta}{2} - \mathrm{sh}\frac{\eta}{2} & 0 \\ 0 & 0 & 0 & \mathrm{ch}\frac{\eta}{2} + \mathrm{sh}\frac{\eta}{2} \end{pmatrix}$$

2. From the operator expression of the energy-momentum four vector

$$P^\mu \equiv \int \mathrm{d}\mathbf{p}\;p^\mu\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}} \qquad\qquad p^\mu = \left(\sqrt{\mathbf{p}^2 + m^2}\,,\,\mathbf{p}\right)$$

and from the canonical commutation relations

$$[\,a_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\,] = \delta(\mathbf{k}-\mathbf{p}) \qquad\qquad [\,a_{\mathbf{k}}\,,\,a_{\mathbf{p}}\,] = 0 = [\,a^\dagger_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\,]$$

one obtains

$$\begin{aligned}
[\,\mathrm{N}\,,\,P^\mu\,] &= \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p}\; p^\mu\, [\,a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\,] \\
&= \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p}\; p^\mu \left(a^\dagger_{\mathbf{k}}\, [\,a_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\,] + [\,a^\dagger_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\,]\, a_{\mathbf{k}} \right) \\
&= \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p}\; p^\mu \left(a^\dagger_{\mathbf{k}}\, [\,a_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\,]\, a_{\mathbf{p}} + a^\dagger_{\mathbf{p}}\, [\,a^\dagger_{\mathbf{k}}\,,\,a_{\mathbf{p}}\,]\, a_{\mathbf{k}} \right) \\
&= \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p}\; p^\mu \left(a^\dagger_{\mathbf{k}}\, a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}\; a_{\mathbf{k}} \right) \delta(\mathbf{k}-\mathbf{p}) \;=\; 0
\end{aligned}$$

3. The canonical anticommutator at arbitrary points between the free Dirac field and its adjoint can be easily calculated to be

$$\begin{aligned}
S\,(x-y) &= \{\psi\,(x)\,,\,\bar\psi\,(y)\} \\
&= \sum_{\mathbf{p}\,,\,r} \Big[\, u_{\mathbf{p}\,,\,r}\,(x) \otimes \bar u_{\mathbf{p}\,,\,r}\,(y) + v_{\mathbf{p}\,,\,r}\,(x) \otimes \bar v_{\mathbf{p}\,,\,r}\,(y) \Big] \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{p}\, \left(\frac{\not p + M}{2\omega_{\mathbf{p}}}\, \mathrm{e}^{-\,\mathrm{i}p\,(x-y)} - \mathrm{e}^{\mathrm{i}p\,(x-y)}\, \frac{M - \not p}{2\omega_{\mathbf{p}}} \right) \\
&= \frac{1}{(2\pi)^3}\, (i\,\not\partial_x + M) \int \mathrm{d}\mathbf{p}\, \left(\frac{1}{2\omega_{\mathbf{p}}}\, \mathrm{e}^{-\,\mathrm{i}p\,(x-y)} - \mathrm{e}^{\mathrm{i}p\,(x-y)}\, \frac{1}{2\omega_{\mathbf{p}}} \right) \\
&= (i\,\not\partial_x + M) \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \mathrm{e}^{-\,\mathrm{i}p\,(x-y)}\, \delta\,(p^2 - M^2)\, \mathrm{sgn}\,(p_0) \\
&= (i\,\not\partial_x + M)\, \frac{1}{\mathrm{i}}\, D\,(x-y)
\end{aligned}$$

where use has been made of the formulæ

$$\sum_{r=1,\,2} \left\{ \begin{array}{l} u_r\,(\mathbf{p}) \otimes \bar u_r\,(\mathbf{p}) = \not p + M \\ v_r\,(\mathbf{p}) \otimes \bar v_r\,(\mathbf{p}) = \not p - M \end{array} \right. \qquad\qquad (p_0 = \omega_{\mathbf{p}})$$

The canonical anticommutator at arbitrary points is a solution of the Dirac equation which vanishes when $(x-y)$ is a space-like interval with $x^0 \neq y^0$, since we have

$$D\,(x-y) \equiv 0 \qquad \forall\,(x_0 - y_0)^2 < (\mathbf{x}-\mathbf{y})^2 \qquad (x^0 \neq y^0)$$

Instead we find the equal time anticommutator

$$S\,(0, \mathbf{x}-\mathbf{y}) = \gamma^0\, \delta\,(\mathbf{x}-\mathbf{y})$$

because of the limits

$$\lim_{x_0 \to y_0} \boldsymbol{\nabla} D(x-y) = 0 \qquad \lim_{x_0 \to y_0} \frac{\partial}{\partial x_0} D(x-y) = \delta(\mathbf{x}-\mathbf{y})$$

0.15 Wednesday 16 July 2008

1. The Lagrangian is not invariant, owing to the presence of the mass term, so that Noether's theorem yields in this case

$$\partial_\mu J_5^\mu(x) = \partial_\mu\Big(\bar\psi(x)\,\gamma^\mu\gamma_5\,\psi(x)\Big) = \; 2\,i\,M\,\bar\psi(x)\,\psi(x)$$

which means that the axial charge

$$Q_5(t) = \int d\mathbf{x}\;\psi^\dagger(x)\,\gamma_5\,\psi(x)$$

in conserved in time only in the massless case.

2. The transformation law under time reversal of the adjoint spinor is evidently given by

$$\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\mathcal{T}^\dagger = \bar\psi(-t,\mathbf{x})\,\gamma^1\gamma^3$$

Then the transformation law for the axial current bilinear under time reversal becomes

$$\begin{aligned}
\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\gamma^0\gamma_5\,\psi(t,\mathbf{x})\,\mathcal{T}^\dagger \;&= \\
\bar\psi(-t,\mathbf{x})\,\gamma^1\gamma^3\gamma^0\gamma_5\,\left(-\gamma^1\gamma^3\right)\,\psi(-t,\mathbf{x}) \;&= \\
(+1)\,\psi^\dagger(-t,\mathbf{x})\,\gamma_5\,\psi(-t,\mathbf{x}) & \\
\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\gamma^1\gamma_5\,\psi(t,\mathbf{x})\,\mathcal{T}^\dagger \;&= \\
\bar\psi(-t,\mathbf{x})\,\gamma^1\gamma^3\gamma^1\gamma_5\,\left(-\gamma^1\gamma^3\right)\,\psi(-t,\mathbf{x}) \;&= \\
(-1)\,\psi^\dagger(-t,\mathbf{x})\,\gamma^1\gamma_5\,\psi(-t,\mathbf{x}) & \\
\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\gamma^2\gamma_5\,\psi(t,\mathbf{x})\,\mathcal{T}^\dagger \;&= \\
\bar\psi(-t,\mathbf{x})\,\gamma^1\gamma^3(\gamma^2)^*\gamma_5\,\left(-\gamma^1\gamma^3\right)\,\psi(-t,\mathbf{x}) \;&= \\
(-1)\,\psi^\dagger(-t,\mathbf{x})\,\gamma^2\gamma_5\,\psi(-t,\mathbf{x}) & \\
\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\gamma^3\gamma_5\,\psi(t,\mathbf{x})\,\mathcal{T}^\dagger \;&= \\
\bar\psi(-t,\mathbf{x})\,\gamma^1\gamma_5\,\left(\gamma^1\gamma^3\right)\,\psi(-t,\mathbf{x}) \;&= \\
(-1)\,\psi^\dagger(-t,\mathbf{x})\,\gamma^3\gamma_5\,\psi(-t,\mathbf{x}) &
\end{aligned}$$

so that we can eventually write

$$\begin{aligned}
\mathcal{T}\,\bar\psi(t,\mathbf{x})\,\gamma^\mu\gamma_5\,\psi(t,\mathbf{x})\,\mathcal{T}^\dagger \;&=\; g^{\mu\nu}\,\bar\psi(-t,\mathbf{x})\,\gamma^\nu\gamma_5\,\psi(-t,\mathbf{x}) \\
&= \begin{cases} \psi(-t,\mathbf{x})^\dagger\,\gamma_5\,\psi(-t,\mathbf{x}) & \text{for } \mu = 0 \\ -\,\bar\psi(-t,\mathbf{x})\,\gamma^\jmath\gamma_5\,\psi(-t,\mathbf{x}) & \text{for } \mu = \jmath = 1,2,3 \end{cases}
\end{aligned}$$

The transformation rules under parity are

$$\mathcal{P}\,\psi\,(t,\mathbf{x})\,\mathcal{P} = \gamma^0\,\psi\,(t,-\,\mathbf{x}) \qquad \mathcal{P}\,\bar{\psi}\,(t,\mathbf{x})\,\mathcal{P} = \bar{\psi}\,(t,-\,\mathbf{x})\,\gamma^0$$

thus we immediately obtain

$$\mathcal{P}\,\bar{\psi}\,(t,\mathbf{x})\,\gamma^\mu\,\gamma_5\,\psi\,(t,\mathbf{x})\,\mathcal{P} \;=\; -\,g^{\mu\nu}\,\bar{\psi}\,(t,-\,\mathbf{x})\,\gamma^\nu\,\gamma_5\,\psi\,(t,-\,\mathbf{x})$$

which is the expected transformation law for a pseudo−vector. The charge conjugation gives

$$\psi^{\,c}\,(x) = \big(-\,i\,\bar{\psi}\,(x)\,\gamma^0\,\gamma^2\big)^\top \qquad \bar{\psi}^{\,c}\,(x) = \big(-\,i\,\gamma^0\,\gamma^2\,\psi\,(x)\big)^\top$$

In order to obtain the transformation law of the axial current under charge conjugation it is very convenient to write down explicitly the spinor indices and to suitably take into account the Grassmann like nature of the classical spinor fields. Hence, since γ^0 and γ_5 are real symmetric matrices in the Weyl representation, we obtain

$$\begin{aligned}
& \bar{\psi}^{\,c}\,(x)\,\gamma^0\,\gamma_5\,\psi^{\,c}\,(x) \\
=\;& -\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x)\,\gamma^0_{\alpha\eta}\,\gamma_{5\,\eta\upsilon}\,\bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
=\;& -\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x)\,\gamma^0_{\eta\alpha}\,\gamma_{5\,\upsilon\eta}\,\bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
=\;& \bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon}\,\gamma_{5\,\upsilon\eta}\,\gamma^0_{\eta\alpha}\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x) \\
=\;& \bar{\psi}\,(x)\,\gamma^0\,\gamma^2\,\gamma_5\,\gamma^0\,\gamma^0\,\gamma^2\,\psi\,(x) \\
=\;& \bar{\psi}\,(x)\,\gamma^0\,\gamma^2\,\gamma_5\,\gamma^2\,\psi\,(x) \\
=\;& \bar{\psi}\,(x)\,\gamma^0\,\gamma_5\,\psi\,(x)
\end{aligned}$$

Furthermore, taking into account that γ^1 and γ^3 are real antisymmetric matrices in the Weyl representation, we get

$$\begin{aligned}
& \bar{\psi}^{\,c}\,(x)\,\gamma^\jmath\,\gamma_5\,\psi^{\,c}\,(x) \\
=\;& -\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x)\,\gamma^\jmath_{\alpha\eta}\,\gamma_{5\,\eta\upsilon}\,\bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
=\;& \gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x)\,\gamma^\jmath_{\eta\alpha}\,\gamma_{5\,\upsilon\eta}\,\bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
=\;& -\,\bar{\psi}_\theta\,(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon}\,\gamma_{5\,\upsilon\eta}\,\gamma^\jmath_{\eta\alpha}\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta\,(x) \\
=\;& -\,\bar{\psi}\,(x)\,\gamma^0\,\gamma^2\,\gamma_5\,\gamma^\jmath\,\gamma^0\,\gamma^2\,\psi\,(x) \\
=\;& \bar{\psi}\,(x)\,\gamma^2\,\gamma_5\,\gamma^\jmath\,\gamma^2\,\psi\,(x) \\
=\;& \bar{\psi}\,(x)\,\gamma^\jmath\,\gamma_5\,\psi\,(x) \qquad \jmath = 1, 3
\end{aligned}$$

Finally, from the symmetry property $\gamma^{2\,\top} = \gamma^2$ we find

$$\bar{\psi}^{\,c}\,(x)\,\gamma^2\,\gamma_5\,\psi^{\,c}\,(x)$$

$$
\begin{aligned}
&= -\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x)\,\gamma^2_{\alpha\eta}\,\gamma_{5\,\eta\upsilon}\,\bar{\psi}_\theta(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
&= -\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x)\,\gamma^2_{\eta\alpha}\,\gamma_{5\,\upsilon\eta}\,\bar{\psi}_\theta(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon} \\
&= \bar{\psi}_\theta(x)\,\gamma^0_{\theta\tau}\,\gamma^2_{\tau\upsilon}\,\gamma_{5\,\upsilon\eta}\,\gamma^2_{\eta\alpha}\,\gamma^0_{\alpha\beta}\,\gamma^2_{\beta\delta}\,\psi_\delta(x) \\
&= \bar{\psi}(x)\,\gamma^0\,\gamma^2\,\gamma_5\,\gamma^2\,\gamma^0\,\gamma^2\,\psi(x) \\
&= \bar{\psi}(x)\,\gamma^0\,\gamma^2\,\gamma_5\,\gamma^0\,\psi(x) \\
&= \bar{\psi}(x)\,\gamma^2\,\gamma_5\,\psi(x)
\end{aligned}
$$

Thus we have definitely proven that the axial current is invariant under charge conjugation, *i.e.*

$$
\mathcal{C}\,\bar{\psi}(x)\,\gamma^\mu\,\gamma_5\,\psi(x)\,\mathcal{C} \;=\; \bar{\psi}^{\,c}(x)\,\gamma^\mu\,\gamma_5\,\psi^{\,c}(x) \;=\; \bar{\psi}(x)\,\gamma^\mu\,\gamma_5\,\psi(x)
$$

As a consequence we eventually come to the conclusion that the axial current turns out to be $\mathcal{C}\,\mathcal{P}\,\mathcal{T}-$odd, that is

$$
\bar{\psi}(x)\,\gamma^\mu\,\gamma_5\,\psi(x) \quad \stackrel{\mathcal{C}PT}{\rightarrow} \quad -\,\bar{\psi}(-\,x)\,\gamma^\mu\,\gamma_5\,\psi(-\,x)
$$

0.16 Friday 6 June 2008

1. The abelian real massive vector field has three physical polarizations while the massless gauge field has two. Since $n = \dim G = 8\times3\times1 = 24$ the number of real independent components of the non$-$abelian real vector field is 72 in the massive case and 48 in the massless case.

   A Dirac bispinor describing massive particles and antiparticles has eight real components for each generator of the internal symmetry Lie algebra $\mathcal{G}$. Hence the number of real independent components is $8 \times 24 = 192$.

2. (i) The Feynman rules give at once the lowest order $O(\alpha)$ amplitude

$$\mathrm{i}\,\mathcal{M} \;=\; \bar{v}_{r'}(p')\,\gamma^{\mu}\,u_r(p)\,\frac{\mathrm{i}\,e^2\,g_{\mu\nu}}{k^2}\;\bar{u}_s(q)\,\gamma^{\nu}\,v_{s'}(q')$$

   where $p + p' = k = q + q'$ is the virtual photon energy-momentum such that $k^2 > 0$. To compute the differential cross section we need an expression for the square modulus of the above amplitud : we find

$$\left(\bar{v}\,\gamma^{\lambda}\,u\right)^* = u^{\dagger}\,\gamma^{\lambda\dagger}\,\gamma^{0\dagger}\,v = u^{\dagger}\,\gamma^0\,\gamma^{\lambda}\,\left(\gamma^0\right)^2\,v = \bar{u}\,\gamma^{\lambda}\,v$$

   that vindicates the great advantage of the adjoint spinor notation. Thus the squared matrix element becomes

$$\begin{aligned} |\,\mathcal{M}\,|^2 \;&=\; \left(\frac{e^2}{k^2}\right)^2 \quad g_{\mu\nu}\,g_{\rho\sigma} \\ &\times\; \Big(\bar{v}_{r'}(p')\,\gamma^{\mu}\,u_r(p)\,\bar{u}_r(p)\,\gamma^{\rho}\,v_{r'}(p')\Big) \\ &\times\; \Big(\bar{u}_s(q)\,\gamma^{\nu}\,v_{s'}(q')\,\bar{v}_{s'}(q')\,\gamma^{\sigma}\,u_s(q)\Big) \end{aligned}$$

   Here we are interested in the squared matrix element averaged over the initial electron and positron polarizations and further summed over the final muon spins

$$\frac{1}{2}\sum_{r=1,2}\,\frac{1}{2}\sum_{r'=1,2}\,\sum_{s=1,2}\,\sum_{s'=1,2}\,|\,\mathcal{M}\,(r,r' \;\rightarrow\; s,s')\,|^2$$

   By making use of the completeness relations

$$\sum_{r=1,2}\left\{\begin{array}{l} u_r(\mathbf{p})\otimes\bar{u}_r(\mathbf{p}) = \not{p} + m_e \\ v_r(\mathbf{p})\otimes\bar{v}_r(\mathbf{p}) = \not{p} - m_e \end{array}\right. \qquad \left(p_0 = \sqrt{\mathbf{p}^2 + m_e^2}\right)$$

$$\sum_{s=1,\,2} \left\{ \begin{array}{l} u_s(\mathbf{q}) \otimes \bar{u}_s(\mathbf{q}) = \not{q} + m_\mu \\ v_s(\mathbf{q}) \otimes \bar{v}_s(\mathbf{q}) = \not{q} - m_\mu \end{array} \right. \qquad \left(q_0 = \sqrt{\mathbf{q}^2 + m_\mu^2} \right)$$

we readily arrive to

$$\frac{1}{4} \sum_{r,r'} \sum_{s,s'} |\mathcal{M}|^2 = \frac{1}{4} \left(\frac{e^2}{k^2 + i\varepsilon} \right)^2 g_{\mu\nu}\, g_{\rho\sigma} \times$$
$$\mathrm{tr}\ [(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\rho]\ \mathrm{tr}\ [(\not{q} + m_\mu)\gamma^\nu(\not{q}' - m_\mu)\gamma^\sigma]$$

To calculate traces of products of gamma matrices, the general method consists of successive displacements of identical matrix−four−vectors. In particular, the trace of an odd number of gamma matrices does vanish, while we easily find

$$\begin{aligned} \mathrm{tr}\ (\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} = 4\,g^{\mu\nu} \\ \mathrm{tr}\ \left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right) &= 4\left(g^{\kappa\lambda}\,g^{\mu\nu} - g^{\kappa\mu}\,g^{\lambda\nu} + g^{\kappa\nu}\,g^{\lambda\mu}\right) \end{aligned}$$

Hence the e^+e^- trace is

$$4\left[p'^\mu p^\rho + p'^\rho p^\mu - g^{\mu\rho}\left(p \cdot p' + m_e^2\right)\right]$$

and similarly the muon pair trace yields

$$4\left[q'^\sigma q^\nu + q'^\nu q^\sigma - g^{\nu\sigma}\left(q \cdot q' + m_\mu^2\right)\right]$$

After contractions of the Lorentz indices we come to the expression

$$\frac{1}{4} \sum_{r,r'} \sum_{s,s'} |\mathcal{M}|^2 = \frac{8e^4}{(k^2)^2} \times$$
$$\Big[(p\cdot q)(p'\cdot q') + (p\cdot q')(p'\cdot q) + (p\cdot p')\,m_\mu^2$$
$$+\ (q\cdot q')\,m_e^2 + 2m_\mu^2 m_e^2\Big]$$

Neglecting the electron mass, in the center of momentum frame of the e^+e^- and $\mu^+\mu^-$ pairs we have for $m_e \approx 0$

$$\begin{array}{ll} \text{electron}: & \mathbf{p} \qquad p^0 = \sqrt{\mathbf{p}^2 + m_e^2} \approx |\mathbf{p}| \\ \text{positron}: & \mathbf{p}' = -\,\mathbf{p} \qquad p_0' = p_0 \\ \text{muon}: & \mathbf{q}\,, \quad q_0 = \sqrt{\mathbf{q}^2 + m_\mu^2} = E \\ \text{antimuon}: & \mathbf{q}' = -\,\mathbf{q} \quad q_0' = q_0 = p_0 = p_0' = \frac{1}{2}\,E_{\mathrm{CM}} = E \end{array}$$

(ii) The Mandelstam's variables are

$$
\begin{aligned}
s &= (p+p')^2 = (q+q')^2 = 2m_\mu^2 + 2q\cdot q' \approx 2p\cdot p' \\
t &= (p-q)^2 = (p'-q')^2 \approx m_\mu^2 - 2p\cdot q = m_\mu^2 - 2p'\cdot q' \\
u &= (p-q')^2 = (q-p')^2 \approx m_\mu^2 - 2p\cdot q' = m_\mu^2 - 2p'\cdot q
\end{aligned}
$$

with $s+t+u \approx 2m_\mu^2$. Then the main formulæ yield

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} = \frac{1}{(16\pi q_0)^2}\cdot\frac{1}{4}\sum_{r,r'}\sum_{s,s'} |\,\mathcal{M}(s,t,u)\,|^2 \cdot \frac{|\,\mathbf{q}\,|}{|\,\mathbf{p}\,|}
$$

Since we have

$$
\begin{aligned}
k^2 = s = E_{\rm CM}^2 &= 4\,E^2 \\
p\cdot p' \;=\; E^2 + |\,\mathbf{p}\,|^2 &\;\approx\; 2\,E^2 \\
p\cdot q = p'\cdot q' = E\,(\,E - |\,\mathbf{q}\,|\,\cos\theta\,) \\
p\cdot q' = p'\cdot q = E\,(\,E + |\,\mathbf{q}\,|\,\cos\theta\,)
\end{aligned}
$$

where θ is the angle between the directions of the incident e^+e^- pair and the produced $\mu^+\mu^-$ pair in the center of momentume frame, we eventually obtain the differential cross section

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} &= \frac{\alpha^2}{4\,E_{\rm CM}^2}\sqrt{1-\frac{m_\mu^2}{E^2}} \\
&\times \left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]
\end{aligned}
$$

When $m_\mu^2 \ll s$ the leading term is provided by

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \approx \frac{\alpha^2}{4\,E_{\rm CM}^2}\,(1+\cos^2\theta)
$$

and integrating over the solid angle we get the total cross section

$$
\sigma \;\approx\; \frac{4\pi\,\alpha^2}{3\,E_{\rm CM}^2}
$$

(iii) The energy dependence of the $e^+e^- \to \mu^+\mu^-$ cross−section sets the scale for all e^+e^- annihilation processes through a virtual photon and consequent production of spin 1/2 point-like fermion pairs

$$
e^+e^- \;\to\; \gamma^* \;\to\; f\bar{f}
$$

At the center of mass square energy and in the ultrarelativistic regime it is given by

$$\sigma \overset{\beta \to 1}{\sim} N_c\, Q_f^2\, \frac{4\pi}{3} \left(\frac{\alpha}{E_{\rm CM}} \right)^2 = N_c\, Q_f^2\, (\hbar\, c)^2\, \frac{86.8 \text{ nanobarns}}{(\, E_{\rm CM} \text{ in GeV})^2}$$
$$= N_c\, Q_f^2 \cdot 1 \text{ unit of R} \qquad (2)$$

where $e\, Q_f$ is the fermion charge while N_c is one for leptons and three for quarks, because each quark in the Standard Model appears in three colors. Asymptotically we expect

$$\sigma\,(\,e^+e^- \;\to\; \text{hadrons}\,) \overset{\beta \to 1}{\sim} 3 \cdot \Big(\sum_{\text{flavours}} Q_f^2 \Big)\ \text{R}$$

where the sum runs over all quarks, the masses of which are smaller than $\sqrt{s}\,/\,2$. Hence, for $s > 4m_t^2$ we expect

$$\sigma\,(\,e^+e^- \;\to\; \text{hadrons}\,) \overset{\beta \to 1}{\sim} 3 \cdot \left(3 \cdot \frac{4}{9} + 3 \cdot \frac{1}{9} \right) \text{R} = 5\ \text{R}$$